Alexander D. Bruno

Local Methods in Nonlinear Differential Equations

Sorubger-Verkag

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Part I
The Local Method of Nonlinear Analysis of Differential Equations

Part II
The Sets of Analyticity of a Normalizing Transformation

Translated from the Russian
by William Hovingh and Courtney S. Coleman

With 94 Figures

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Introduction to the English Edition by Dr. Steven Wiggins

When faced with a new problem, an often effective strategy for solving it is to somehow transform it into the form of a problem previously solved. Obviously there are no general recipes for implementing such a procedure; instead one must usually rely on insight and luck.

However, in the context of the "local" theory of ordinary differential equations ("local" means in a neighborhood of some invariant manifold) such a procedure for reducing an ordinary differential equation to the "simplest possible form" (what this phrase means can be precisely defined and depends on the particular problem) or normal form is available and goes by the name of the method of normal forms.

The method of normal forms is not new and is usually attributed to Poincaré although some of the basic ideas of the method can be found in earlier works of Jacobi, Briot and Bouquet. Following Poincaré, the method was further developed and utilized in the works of Lyapunov, Dulac, Birkhoff, Cherry, Siegel, Moser, Sternberg, Chen, and Gustavson.

In this book, A. D. Bruno gives an account of the work of these mathematicians as well as the results of his own extensive investigations on the subject. The book begins with a thorough development of the analytical techniques necessary for the implementation of the theory as well as an extensive description of the geometry of the Newton polygon. He then proceeds to discuss the normal form of systems of ordinary differential equations giving many specific applications of the theory. An underlying theme of the book is the unifying nature of the method of normal forms regarding techniques for the study of the local properties of ordinary differential equations. Indeed, Bruno shows that many of the techniques used for studying ordinary differential equations having a small parameter (e.g. the method of averaging) may be viewed as a special case of the method of normal forms. In the second part of the book he shows, for a special class of equations, how the method of normal forms yields classical results of Lyapunov concerning families of periodic orbits in the neighborhood of equilibrium points of Hamiltonian systems as well as the more modern results concerning families of quasiperiodic orbits obtained by Kolmogorov, Arnold, and Moser.

One of the more beneficial results of the recent explosion of interest in non-

linear dynamical systems is the increased interaction between mathematicians and applied scientists. Regarding the method of normal forms, this has led to new developments in the theory as well as a variety of applications of the method to bifurcation theory, fluid mechanics, and structural mechanics. The following references supplement those given by Bruno.

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Pasadena, California

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Part II
The Sets of Analyticity of a Normalizing Transformation

Part I
The Local Method of Nonlinear Analysis of Differential Equations

Table of Contents

Basic Notation

$\dot{x} = dx/dt$
$X = (x_1, \dots, x_n) \qquad Y = (y_1, \dots, y_n)$
$\Lambda = (\lambda_1, \dots, \lambda_n) \qquad Q = (q_1, \dots, q_n)$

Scalars are denoted by small letters, vectors and matrices by capitals (only in Chapter V are capital letters used for scalars). Vectors are written as rows, but are considered to be columns in matrix multiplication.

$\langle Q, \Lambda \rangle = q_1\lambda_1 + q_2\lambda_2 + \cdots + q_n\lambda_n$
$X^Q = x_1^{q_1} x_2^{q_2} \dots x_n^{q_n}$
$\|Q\| = q_1 + q_2 + \cdots + q_n$
$|Q| = |q_1| + |q_2| + \cdots + |q_n|$
$|X| = (|x_1|, |x_2|, \dots, |x_n|)$
E_k—the k^{th} unit vector.
$E^{(m)}$—the $m \times m$ identity matrix.
$V \geqslant 0$ means that $v_1 \geqslant 0, \dots, v_n \geqslant 0$
$\bar{\lambda}$—the complex conjugate of λ.
$\phi_i(X) = x_i f_i(X) \equiv x_i \sum f_{iQ} X^Q$,
$F_Q = (f_{1Q}, f_{2Q}, \cdots, f_{nQ})$,
$F = (f_1, \dots, f_n) \equiv \sum F_Q X^Q$.
$\mathbf{R}_1^n$—the space of exponents
$\mathbf{R}_2^n$—the conjugate (dual) space to $\mathbf{R}_1^n$. Sets in these spaces are denoted by boldface Latin capitals (**L**, **M**, ...) and by the letters Γ, Δ.
L—a linear subspace of $\mathbf{R}_1^n$ consisting of all real solutions Q of the equation $\langle Q, \Lambda \rangle = 0$.
$\mathbf{O} = \{Q: Q \geqslant 0\}$—the non-negative orthant (quadrant, octant, ...)
$\mathbf{Z}^n$—the integer lattice in $\mathbf{R}_1^n$.
$\mathbf{D} = \mathbf{D}(f) = \{Q: f_Q \neq 0\}$—the support of the series $f = \sum f_Q X^Q$.
Δ—the convex hull of the set **D**.
Γ—the closure of Δ.
$\partial\Gamma$—the boundary of Γ.
$\Gamma_j^{(d)}$—a face of the polyhedron Γ.
$\hat{\Gamma}$—Newton open polygon, a part of the boundary $\partial\Gamma$.

$\mathbf{V}_j^{(d)}$—the tangent cone to the face $\Gamma_j^{(d)}$.
$\mathbf{U}_j^{(d)}$—the normal cone to $\Gamma_j^{(d)}$.
$\mathbf{U}_j^{(d)}(\varepsilon)$—a set in $\mathbf{R}_2^n$, corresponding to the face $\Gamma_j^{(d)}$ and to the number $\varepsilon > 0$.
$\mathbf{R}_0^n$ or $\mathbf{C}_0^n$—the basic space with coordinates X, Y or $Z, \ldots$

Capital script letters denote sets in $\mathbf{R}_0^n$ (or $\mathbf{C}_0^n$) and classes of functions and power series:
$\mathcal{U}$—a neighborhood of the point $X = 0$,
$\mathcal{U}_j^{(d)}(\varepsilon)$—a part of $\mathcal{U}$, corresponding to the face $\Gamma_j^{(d)}$,
$\mathcal{V}_j^{(d)}$—the class of power series, the supports of which lie in the set $\mathbf{V}_j^{(d)}$.

A tilde is placed below the capital Greek letters, $\underset{\sim}{Z}$, $\underset{\sim}{H}$, $\underset{\sim}{M}$, $\underset{\sim}{N}$, $\underset{\sim}{E}$ to distinguish them from the Latin letters, Z, H, M, N, E, which are similar. All asterisks have the same meaning, regardless of type size.

Introduction

1. Classification of Equations

In many cases, problems in mechanics, physics, and other natural sciences can be reduced to the solution of a system of equations

$$f_j(x_1, \dots, x_n) = 0 \ , \qquad j = 1, \dots, m \ , \tag{1}$$

where the f_j are polynomials in the variables,

$$f_j = \sum a_{j_{q_1 q_2 \cdots q_n}} x_1^{q_1} x_2^{q_2} \dots x_n^{q_n} \ , \qquad j = 1, \dots, m \ , \tag{2}$$

or else are close to polynomials. Depending on the nature of the variables x_i and coefficients a, different types of equations are obtained—types which are studied by different mathematical sciences. Generally speaking, the classification of these types takes the following form:

I. The coefficients a are integers or rational numbers, as are the desired solutions x_i. These are the problems of number theory.

II. The coefficients a and solutions x_i are real or complex numbers. These problems arise in algebra—specifically, in algebraic geometry and in the theory of analytic sets.

III. The coefficients a are real or complex functions of some variable t, the unknowns are functions of this variable, $x_1 = y_1(t), \dots, x_l = y_l(t)$ and derivatives of these functions, $x_j = d^s y_i / dt^s (j > l)$. The problem consists of seeking those functions $y_1(t), \dots, y_l(t)$ which satisfy system (1). Such problems are treated by the theory of ordinary differential equations.

IV. The coefficients a are functions of several variables $t_1, \dots, t_k$, the unknowns are functions of these variables, $x_1 = y_1(t_1, \dots, t_k), \dots, x_l = y_l(t_1, \dots, t_k)$ and their derivatives:

$$x_j = \frac{\partial^s y_i}{\partial t_1^{s_1} \partial t_2^{s_2} \dots \partial t_k^{s_k}} \ , \qquad j > l, s = s_1 + \cdots + s_k \ .$$

The problem of finding functions y_i which satisfy (1) belongs to the theory of partial differential equations.

Each type can be further divided into linear and non-linear problems. System (1) is called *linear* if each of the variables x_i appears in the polynomials of (2) only in the first or the zeroth power; that is, if, in each term of the polynomial of (2), the q_k are either zero or one and their sum is not more than one. If this is not the case, then system (1) is *non-linear*. Linear problems are normally much easier to solve than non-linear ones.

Further, we shall distinguish between local and global problems. A *local problem* consists of investigating those solutions X of system (1) which lie in an arbitrarily small neighborhood of some known solution X^0, for one or another definition of neighborhood. It is important that the size of the neighborhood not be essential; it is enough to find all solutions in some neighborhood of the solution X^0. A *global problem*, on the other hand, consists of seeking all solutions to system (1) which belong to some fixed set—for example, for a problem of the second type finding all complex solutions X. Global problems are usually more difficult than local problems.

To date some general approaches to non-linear problems have been developed, particularly in the local case. Non-linear problems are identified as being either quasi-linear or essentially non-linear. *Quasi-linear problems* are solved either by taking into account nonlinear "perturbations" (the classical approach) or through a change of coordinates which simplifies (1) and transforms it into a so-called "normal form" (this is the modern approach). In *essentially non-linear problems*, one must know how to choose a first approximation to system (1) (i.e., a truncation) that leads to linear inequalities in the exponents $q_1, \dots, q_n$ of expression (2). These inequalities are solved by means of a geometrical construction ("Newton's polyhedron"). By means of power transformations, non-linear in X but linear with respect to the q_i and $\ln x_i$, one can succeed in simplifying the truncation. By this means, one can reduce an essentially non-linear problem to several quasi-linear problems.

Remarkably, essentially non-linear problems are thus closely connected with certain linear geometry in the "space of the exponents."

2. Fundamental Ideas of the Local Method

We will explain these ideas by considering those solutions of a system of ordinary differential equations

$$dX/dt = \Phi(X) \tag{3}$$

which lie in a neighborhood $\mathscr{U}$ of the stationary solution $X = (x_1, \dots, x_n) = 0$. We consider in parallel the cases of real and complex X and Φ. Here, the function $\Phi(x)$ is assumed to be analytic or sufficiently smooth in the neighborhood $\mathscr{U}$. We make no further restrictions: we even consider cases of degeneracy and of resonance.

We shall solve the problem posed above by constructing a special local coordinate transformation which will change (3) into an integrable system. Such

a transformation exists for the entire neighborhood only in comparatively simple cases. In complicated cases, the neighborhood being investigated must be divided up into parts, each with its own local coordinates with respect to which the system is integrable. Generally speaking, the construction of these coordinate changes and decompositions is accomplished step by step, and not all at once. At each step we construct a finer decomposition, and in each part of the neighborhood we introduce a coordinate system within which the system is simpler. Working under very general assumptions, we can arrive after a finite number of steps at a system which is integrable in each part of the neighborhood.

In systems arising in mechanical, physical, and astronomical problems it is usually sufficient to go through one (occasionally two) such steps. The foundation of this local method rests on two basic ideas:

1. *The normal form.* If system (3) has a non-zero linear part, then we can transform it into a *normal form*, which easily reduces to a system of lesser order with a vanishing linear part, and is often immediately integrable. Various cases of such a normalizing transformation have been considered by Poincaré, Picard, Horn, Dulac, Birkhoff, Cherry, Siegel, Moser, Sternberg, and others. A definitive form, a method of reduction of order and general properties of the normal form have been described by the author [Bruno 1964–1976].

2. *The reduction of complex singularities.* If the expansion of the vector-function $\Phi(X)$ in powers of the x_i contains no linear terms, then we may select a finite number of "truncated systems"

$$dX/dt = \hat{\Phi}_j^{(d)}(X) \tag{4}$$

and correspondingly divide the neighborhood $\mathcal{U}$ into parts $\mathcal{U}_j^{(d)}(\varepsilon)$ such that each part $\mathcal{U}_j^{(d)}(\varepsilon)$ is a curvilinear cone touching the stationary point, while the truncation (4) is the first non-trivial approximation to system (3) in $\mathcal{U}_j^{(d)}(\varepsilon)$.

This is effected by the construction of a certain polyhedron in the exponent space. Further, for a simple (in some sense) truncation (4), we can construct a normal form of system (3) in the part $\mathcal{U}_j^{(d)}(\varepsilon)$—that is, we can simplify system (3) in that region. For a more complicated truncation (4) we must make a birational (power) transformation, blowing up the stationary point into a manifold $\mathcal{M}$ and the region $\mathcal{U}_j^{(d)}(\varepsilon)$ into a neighborhood $\mathcal{U}'$ of this manifold. It is then necessary to find the stationary points in $\mathcal{M}$ and to study their neighborhoods, which will be parts of the neighborhood $\mathcal{U}'$. In each of these parts the system will be in some sense simpler than the initial system, and we can employ anew our simplifying and reducing constructions. This method of reducing singularities is analogous to the "multiple sigma-process" of algebraic geometry and has its origin in a Newton polygon. For a two-dimensional system of differential equations (3), various truncated systems (4) have been selected and studied by Briot and Bouquet [Briot, Bouquet, 1856], Horn, Frommer, and others. This has been done for multi-dimensional systems by the author [Bruno, 1962, 1965].

The local method, naturally, has two parts. The first—the algebraic part—consists of finding an algorithm for constructing the required formal power

series. The second part consists of interpreting these series by means of analytic functions (Poincaré, Dulac, Siegel, Pliss, and the author [Bruno, 1971]) or smooth functions (Birkhoff, Sternberg, Hartmann, Grobman, and others), or in estimating the accuracy of approximate integration by means of these methods (Birkhoff, Siegel, Moser, and others).

3. The Contents of this Monograph

The main objects of study of the present work are local non-linear problems of the second and third types, though their study sometimes reduces to that of linear problems of the first or fourth types. In addition, our attention is primarily directed towards analytic systems (1), where the f_j are analytic functions of their arguments.

Chapters I and II consider two-dimensional systems in great detail. Chapters III and IV treat multi-dimensional systems in slightly less detail. Chapter V applies the local method to concrete mechanical problems.

§ 1 of Chapter I is devoted to the solution of linear inequalities, and in connection with this discusses geometry in the plane $\mathbf{R}_1^2$ and the dual plane $\mathbf{R}_2^2$. The techniques developed there will be the basis of the "geometry of exponents". § 2 of Chapter I presents methods of solving an analytic equation

$$f(x_1, x_2) = 0$$

in the neighborhood of the critical point $x_1 = x_2 = 0$. § 3 applies the local method to the study of the level curve $f(x_1, x_2) = c$ in the neighborhood of the critical point. This is the simplest problem with which we can demonstrate the whole apparatus of the local method.

Chapter II treats the system of differential equations

$$dx_1/dt = \varphi_1(x_1, x_2) \ ,$$

$$dx_2/dt = \varphi_2(x_1, x_2) \ ,$$

in the neighborhood of a singular point using the local method.

Chapter III systematically presents the theory of the normal form of the system of differential equations (3) in a neighborhood of the elementary singular point $X = 0$. § 4 of the chapter extends this theory to a neighborhood of an invariant manifold.

§ 1 of Chapter IV treats the application of the local method to the solution of system (3) in the neighborhood of a non-elementary singular point. § 2 gives a review of recent investigations using the geometry of exponents (Newton's polygon or polyhedron).

§ 1 of Chapter V considers the problem of the influence of nutational oscillations on the rate of precession of a heavy gyroscope in a Cardan suspension. This example illustrates the ideas of §§ 1, Chapters II and III. § 2 of Chapter V

treats the problem of the existence of periodic oscillations of a satellite in the plane of an elliptical orbit; this illustrates the methods of §§3 and 4 of Chapter III.

The treatment of Chapters I, II, and V and §§1, 2 of Chapter III is detailed (especially Chapters I and II). The last half of Chapter III and all of Chapter IV, in contrast, are more review and problem oriented. Chapters I and II are based on lectures given by the author [Bruno, 1973a] and are written in the form of a text book. These chapters contain detailed proofs, many figures, examples, and exercises. They should be understandable to students who have completed two years of university study. Chapters III and IV are based on work by the author [Bruno, 1971, 1972a] and a series of his articles. The presentation is such that it should be possible to read each chapter independently. For a short outline of the local method in the author's own translation see the author's lectures [Bruno, 1987].

A recent development of the local method can be found in the following works by the author:

On stability in a Hamiltonian system. Mat. Zametki, 40, No. 3, 385–392 (1986) [Russian]. Math. Notes, 40, No. 3, 726–730 (1986) [English]

Bifurcations of periodic solutions in a symmetrical case of a multiple pair of imaginary eigenvalues. In: The Numerical Solution of Ordinary Differential Equations. Akad Nauk SSSR Inst. Prikl. Mat., 161–176, 239 (1988) [Russian]

The normal form of a Hamiltonian system. Usp. Mat. Nauk, 43, No. 1, 23–56 (1988) [Russian]. Russ. Math. Surv., 43, No. 1 (1988) [English]

The normalization of a Hamiltonian system near a cycle or a torus. Usp. Mat. Nauk, 44, No. 1 (1989) [Russian]. Russ. Math. Surv., 44, No. 1 (1989) [English]

On small divisors. Banach Center Publications. Warsawa: PWN, v. 23 (1989) [English]

On the question of stability in a Hamiltonian system. Ibid.

[Translator's note: the author's name is often spelled "Bryuno" in translations, but "Bruno" is the spelling used throughout this monograph.]

Chapter I
Foundations of the Local Method

§ 1. Linear Inequalities in the Plane

1.1. Basic Definitions

By $\mathbf{R}^2$ we will denote a real two-dimensional vector space. That is, $\mathbf{R}^2$ is the real plane, with Cartesian coordinates q_1 and q_2. The points of this plane are two-dimensional vectors $Q = (q_1, q_2)$. If p_1, p_2 and c are fixed real numbers, then the solutions of the equation

$$q_1 p_1 + q_2 p_2 = c \tag{1}$$

lie along a straight line in $\mathbf{R}^2$ perpendicular to the vector $P = (p_1, p_2)$.

Now let us vary the magnitude of c and consider the lines defined by (1) for different values of c. These will be mutually parallel lines. If $c = 0$, the line will pass through the origin, $Q = 0$; if $c > 0$, the line will be displaced in the direction of P, and the displacement will increase with increasing c; finally, when $c < 0$, the line will be displaced in the opposite direction (figure 1). Each of these lines divides the plane into two half-planes: the positive half-plane, in which $q_1 p_1 + q_2 p_2 > c$, and the negative, in which $q_1 p_1 + q_2 p_2 \leqslant c$. Note that if we use the scalar product $\langle Q, P \rangle = q_1 p_1 + q_2 p_2$, then equation (1) can be written in the form

$$\langle Q, P \rangle = c \ .$$

From now on we will consider the vectors P and Q, the two factors in this scalar product, to be points in two distinct planes:

$$Q \in \mathbf{R}_1^2 \ , \qquad P \in \mathbf{R}_2^2 \ .$$

The reason for this separation will become clear as we proceed.

Let Δ be a point set in $\mathbf{R}^2$; it is *convex* if, along with any two of its points Q_1 and Q_2, the set contains the entire line segment joining the two points. The set Δ is called the *convex hull* of a set $\mathbf{D}$ if Δ is the smallest convex set containing $\mathbf{D}$. For example, the convex hull of two points Q_1 and Q_2 is the line segment which joins them; the convex hull of three points is the triangle with its vertices at those three points. This fact can be expressed by the formula

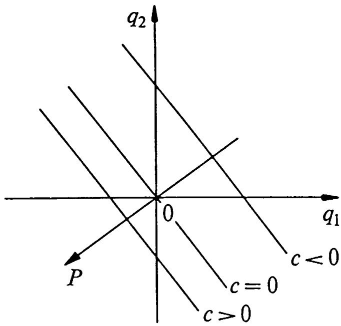

Fig. 1

$$\Delta = \{Q\colon Q = \delta_1 Q_1 + \delta_2 Q_2 + \delta_3 Q_3;$$
$$\delta_1, \delta_2, \delta_3 \geqslant 0;\ \delta_1 + \delta_2 + \delta_3 = 1;\ Q_1, Q_2, Q_3 \in \mathbf{D}\} \ .$$

If $\mathbf{D}$ is a finite set of points, then its convex hull Δ will be a closed polygon whose vertices are points in $\mathbf{D}$. If a set $\mathbf{D}$ contains infinitely many points, its convex hull may not be closed, and may not be a polygon. For instance, let

$$\mathbf{D} = \{Q\colon q_1 = 1/m,\ q_2 = 0;\ m = 1, 2, \dots\} \ , \tag{2}$$

then Δ is the half-open interval $(0, 1]$ on the q_1 axis. A second example: the convex hull of a circle is a disk.

Let the set $\mathbf{D} \subset \mathbf{R}_1^2$ and the vector $P \in \mathbf{R}_2^2$ be fixed, and let $c = \sup\langle Q, P\rangle$ for all $Q \in \mathbf{D}$. Then the equation

$$\langle Q, P\rangle = c$$

determines in the plane $\mathbf{R}_1^2$ the *support line* $\mathbf{L}_P$ of the set $\mathbf{D}$ with respect to the vector P. The inequality

$$\langle Q, P\rangle \leqslant c$$

defines the corresponding *supporting half-plane* $\mathbf{L}_P^{(-)}$. The half-plane $\langle Q, P\rangle > c$ contains no points of $\mathbf{D}$; however, any half-plane determined by

$$\langle Q, P\rangle > c - \varepsilon \ , \qquad \varepsilon > 0$$

does contain points of the set $\mathbf{D}$. If $\mathbf{D}$ consists of a finite number of points Q, then the supremum of the scalar products $\langle Q, P\rangle$ is equal to the maximum, and some points of $\mathbf{D}$ lie on the support line $\mathbf{L}_P$. The support line for an infinite set $\mathbf{D}$ may not contain points of $\mathbf{D}$ (for example, the set (2), where $P = (-1, 0)$). The intersection

$$\mathbf{D} \cap \mathbf{L}_P = \mathbf{D}_P$$

consists of all points of the set $\mathbf{D}$ for which the scalar product $\langle Q, P\rangle$ has its maximum value.

By Γ let us denote the intersection of all the supporting half-planes of $\mathbf{D}$; that is,

$$\Gamma = \bigcap_{P \neq 0} \mathbf{L}_P^{(-)} \quad \text{for all } P \in \mathbf{R}_2^2 \ .$$

Clearly, Γ is a closed convex set which contains the convex hull Δ of $\mathbf{D}$:

$$\Gamma \supset \Delta \supset \mathbf{D} \ .$$

As long as Γ is a closed set, it contains the closure of Δ.

Theorem 1. *For every set* $\mathbf{D}$, *the intersection* Γ *of all its supporting half-planes coincides with the closure of the convex hull* Δ.

For the proof, see § 1 of Busemann [1958].

We call the intersection of the set Γ with a support line $\mathbf{L}_P$ a *face*. The faces of the set Γ lie on its boundary $\partial\Gamma$; conversely, the boundary $\partial\Gamma$ is composed of the faces of Γ. If Γ is a polygon, then its faces are edges and vertices. We will denote the faces of Γ by $\Gamma_j^{(d)}$, where d is the dimension of the object (i.e. $d = 0$ for a vertex and $d = 1$ for an edge) and j is its number. On each face $\Gamma_j^{(d)}$ we identify a subset of $\mathbf{D}$,

$$\mathbf{D}_j^{(d)} = \Gamma_j^{(d)} \cap \mathbf{D} \ ,$$

consisting of those points of $\mathbf{D}$ which lie on the face $\Gamma_j^{(d)}$. We will call the face $\Gamma_j^{(d)}$ *proper* if it has the same dimension as the convex hull of $\mathbf{D}_j^{(d)}$; otherwise, we call $\Gamma_j^{(d)}$ *improper*.

Example 1. $\mathbf{D} = \{Q_1, Q_2, Q_3\}$, where $Q_1 = (1,0)$, $Q_2 = (0,1)$, and $Q_3 = (1,1)$ (figure 2). The support line for the vector $P_1 = (-1,-1)$ is given by the equation $\langle Q_1, P_1\rangle = -1$ (since $\langle Q_1, P_1\rangle = \langle Q_2, P_1\rangle = -1, \langle Q_3, P_1\rangle = -2 < -1$). Thus,

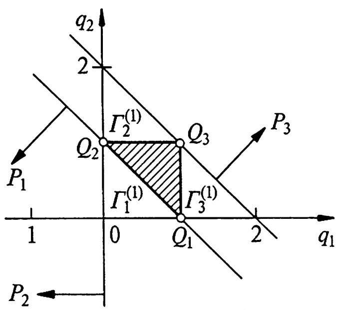

Fig. 2

the supporting half-plane $\mathbf{L}_{P_1}^{(-)}$ is defined by the inequality $-q_1 - q_2 \leqslant -1$. Similarly if $P_2 = (-1, 0)$, the support line $\mathbf{L}_{P_2}$ is given by the equation $q_1 = 0$ and the supporting half-plane $\mathbf{L}_{P_2}^{(-)}$ by the inequality $-q_1 \leqslant 0$. Finally, the support line $\mathbf{L}_{P_3}$ and the supporting half-plane $\mathbf{L}_{P_3}^{(-)}$ for $P_3 = (1, 1)$ are $q_1 + q_2 = 2$ and $q_1 + q_2 \leqslant 2$, respectively. The polygon Γ is the triangle with vertices Q_1, Q_2, Q_3. Thus

$$\Gamma_j^{(0)} = Q_j \ , \qquad j = 1, 2, 3 \ ;$$

the edges $\Gamma_j^{(1)}$ are the line segments connecting Q_j with Q_{j+1}. Correspondingly,

$$\mathbf{D}_j^{(0)} = Q_j \ , \qquad\qquad j = 1, 2, 3 \ ;$$

$$\mathbf{D}_1^{(1)} = \{Q_1, Q_2\} \ , \qquad \mathbf{D}_2^{(1)} = \{Q_2, Q_3\} \ , \qquad \mathbf{D}_3^{(1)} = \{Q_3, Q_1\} \ .$$

Here, all faces are proper.

Example 2. $\mathbf{D}$ is the interior of the unit circle:

$$\mathbf{D} = \{Q: q_1^2 + q_2^2 < 1\} \ .$$

Then $\Delta = \mathbf{D}$, and Γ is the closed unit disk:

$$\Gamma = \{Q: q_1^2 + q_2^2 \leqslant 1\} \ .$$

The boundary $\partial\Gamma$ of Γ is the unit circle $q_1^2 + q_2^2 = 1$. Each of its points is a face $\Gamma_j^{(0)}$. All $\mathbf{D}_j^{(0)}$ are empty (the dimension is -1); therefore all faces are improper.

Note that an edge $\Gamma_j^{(1)}$ will be either a line (thus unique), a half-line (of which there can be no more than two), or a line segment.

1.2. Cones

A set $\mathbf{K} \subset \mathbf{R}^2$ is called a *cone* (see Goldman and Tucker, 1956) if, along with the point P, it contains the ray cP (for all $c > 0$). Thus, the cones in $\mathbf{R}^2$ are: the origin $P = 0$, any ray, any sector, and the whole plane. A cone $\mathbf{K}$ is *convex* if it is a convex set. Thus, a sector is a convex cone only if its interior angle is no larger than π.

Let $\mathbf{L}_P$ be the support line of a set $\mathbf{D}$ for a vector P; then for all $c > 0$, $\mathbf{L}_{cP} = \mathbf{L}_P$. That is, all vectors P' for which $\mathbf{L}_P$ is the support line lie along the ray $\mathbf{P} = \{cP, c > 0\}$.

Let us denote by $\mathbf{U}$ the set of all vectors $P \in \mathbf{R}_2^2$ for which there exists a supporting half-plane $\mathbf{L}_P^{(-)}$ for the set $\mathbf{D}$.

Theorem 2. *The set* $\mathbf{U}$ *is a convex cone.*

Proof. Let two vectors P_1 and P_2 lie in the set $\mathbf{U}$. We will show that $\mathbf{U}$ contains every vector $P = \alpha_1 P_1 + \alpha_2 P_2$, where $\alpha_1, \alpha_2 \geqslant 0$. The set Γ has two

supporting half-planes, $\mathbf{L}_{P_1}^{(-)}$ and $\mathbf{L}_{P_2}^{(-)}$; consequently, it is contained in their intersection:

$$\Gamma \subset \mathbf{L}_{P_1}^{(-)} \cap \mathbf{L}_{P_2}^{(-)} \ . \tag{3}$$

If P_1 and P_2 are not parallel, the support lines $\mathbf{L}_{P_1}$ and $\mathbf{L}_{P_2}$ divide the plane $\mathbf{R}_1^2$ into four parts; one of these is $\mathbf{L}_{P_1}^{(-)} \cap \mathbf{L}_{P_2}^{(-)}$. This set has support lines for all vectors P lying between the rays $\{cP_1\}$ and $\{cP_2\}$. By virtue of (3), Γ also has support lines for all such P.

If P_1 and P_2 are parallel, on the other hand, then they lie either along the same ray or else along the same line. Either of these sets is convex. The theorem is proved. □

We will call the set $\mathbf{U}$ the *normal cone of the set* $\mathbf{D}$ (or of the set Γ).

Let $\Gamma_j^{(d)}$ be a face of some closed convex set Γ. The set $\mathbf{U}_j^{(d)}$ of all vectors $P \in \mathbf{R}_2^2$, for which the support line $\mathbf{L}_P$ intersects Γ precisely at the face $\Gamma_j^{(d)}$ is called the *normal cone of the face* $\Gamma_j^{(d)}$:

$$\mathbf{U}_j^{(d)} = \{P\colon \Gamma_j^{(d)} = \mathbf{L}_P \cap \Gamma\} \ .$$

It is easily seen that $\mathbf{U}_j^{(d)}$ is a convex cone. The union of all the normal cones $\mathbf{U}_j^{(d)}$ of the faces $\Gamma_j^{(d)}$ is the normal cone $\mathbf{U}$ of the whole set Γ. If $\Gamma_j^{(d)} = \mathbf{L}_P \cap \Gamma$ and $\tilde{Q} \in \mathbf{L}_P$, then

$$\begin{cases} \langle Q, P\rangle = \langle \tilde{Q}, P\rangle & \text{for all } Q \in \Gamma_j^{(d)} \ , \\ \langle Q, P\rangle < \langle \tilde{Q}, P\rangle & \text{for all } Q \in \Gamma \backslash \Gamma_j^{(d)} \ . \end{cases} \tag{4}$$

Therefore, the vectors in the normal cone $\mathbf{U}_j^{(d)}$ are determined by the system of equations and inequalities (4). The first of these conditions implies that the line $\langle Q, P\rangle = \langle \tilde{Q}, P\rangle$ contains the face $\Gamma_j^{(d)}$; the second implies that this line is the support line to the set Γ. But then it must be the support line to the set $\mathbf{D}$, so that the second condition of (4) can be rewritten as $\langle Q, P\rangle < \langle \tilde{Q}, P\rangle$ for all $Q \in \mathbf{D} \backslash \mathbf{D}_j^{(d)}$.

If the boundary subset $\mathbf{D}_j^{(d)}$ is non-empty, we can take $\tilde{Q} \in \mathbf{D}_j^{(d)}$. Finally, if $\Gamma_j^{(d)}$ is a proper face, the first condition of (4) need only be applied to the points of $\mathbf{D}_j^{(d)}$. Thus, for some point $\tilde{Q} \in \mathbf{D}_j^{(d)}$

$$\mathbf{U}_j^{(d)} = \left\{P\colon \begin{array}{ll} \langle Q, P\rangle = \langle \tilde{Q}, P\rangle & \text{for all } Q \in \mathbf{D}_j^{(d)}, \\ \langle Q, P\rangle < \langle \tilde{Q}, P\rangle & \text{for all } Q \in \mathbf{D} \backslash \mathbf{D}_j^{(d)} \end{array}\right\} \ . \tag{5}$$

Example 3 (continuation of example 1). $\mathbf{D} = \{\{Q_1, Q_2, Q_3\}; Q_1 = (1, 0), Q_2 = (0, 1), Q_3 = (1, 1)\}$ (figure 2). Here, the cone $\mathbf{U}$ will be the entire plane $\mathbf{R}_2^2$. For the edge $\Gamma_1^{(1)}$, we have (in agreement with (5)) the normal cone

$$\mathbf{U}_1^{(1)} = \left\{P\colon \begin{array}{l} \langle Q_1, P\rangle = \langle Q_2, P\rangle, \\ \langle Q_3, P\rangle < \langle Q_1, P\rangle \end{array}\right\} \ .$$

Taking into account the numerical values of the Q_j, we have

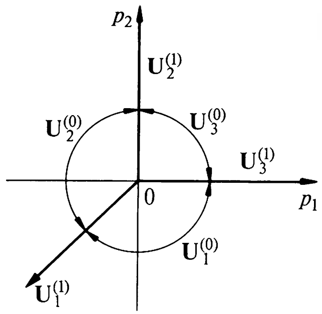

Fig. 3

$$\mathbf{U}_1^{(1)} = \{P: p_1 = p_2, p_1 + p_2 < p_1\}$$

or

$$\mathbf{U}_1^{(1)} = \{P: p_1 = p_2 < 0\} \ .$$

Thus, the normal cone $\mathbf{U}_1^{(1)}$ is the ray in the p_1, p_2 plane which bisects the third quadrant (figure 3). This ray is perpendicular to edge $\Gamma_1^{(1)}$ and is directed out of the polygon Γ. Similarly we find the normal cone

$$\mathbf{U}_2^{(1)} = \left\{P: \begin{matrix} \langle Q_2, P\rangle = \langle Q_3, P\rangle \\ \langle Q_1, P\rangle < \langle Q_2, P\rangle \end{matrix}\right\}$$

$$= \{P: p_1 = 0, p_2 > 0\} \ .$$

In the p_1, p_2 plane the cone $\mathbf{U}_2^{(1)}$ is a ray orthogonal to the edge $\Gamma_2^{(1)}$ (figures 2 and 3). Finally,

$$\mathbf{U}_3^{(1)} = \left\{P: \begin{matrix} \langle Q_3, P\rangle = \langle Q_1, P\rangle \\ \langle Q_2, P\rangle < \langle Q_1, P\rangle \end{matrix}\right\}$$

$$= \{P: p_1 > 0, p_2 = 0\} \ .$$

This is also a ray (figure 3). We will now find the cones of the vertexes. We first note that the vertex $\Gamma_j^{(0)}$ contains a single point of the set **D**, so only the inequalities in (5) are meaningful. Thus, we have the normal cone

$$\mathbf{U}_1^{(0)} = \{P: \langle Q_2, P\rangle < \langle Q_1, P\rangle; \langle Q_3, P\rangle < \langle Q_1, P\rangle\} \ .$$

Taking into account the coordinates of the points Q_j, we obtain

$$\mathbf{U}_1^{(0)} = \{P: p_2 < p_1, p_1 + p_2 < p_1\} = \{P: p_2 < p_1, p_2 < 0\} \ .$$

The normal cone $\mathbf{U}_1^{(0)}$ is thus the sector of the p_1, p_2 plane bounded by the rays

$\mathbf{U}_1^{(1)}$ and $\mathbf{U}_3^{(1)}$ (figure 3). Similarly, we have the normal cone

$$\begin{aligned}\mathbf{U}_2^{(0)} &= \{P: \langle Q_1, P\rangle < \langle Q_2, P\rangle, \langle Q_3, P\rangle < \langle Q_2, P\rangle\} \\ &= \{P: p_1 < p_2, p_1 < 0\} \ .\end{aligned}$$

This is the sector of the p_1, p_2 plane bounded by the rays $\mathbf{U}_1^{(1)}$ and $\mathbf{U}_2^{(1)}$. Finally,

$$\mathbf{U}_3^{(0)} = \{P: \langle Q_1, P\rangle < \langle Q_3, P\rangle, \langle Q_2, P\rangle < \langle Q_3, P\rangle\} = \{P > 0\} \ ,$$

the sector bounded by the rays $\mathbf{U}_2^{(1)}$ and $\mathbf{U}_3^{(1)}$ (figure 3).

Example 4 (continuation of example 2). Again, $\mathbf{U}$ is the plane $\mathbf{R}_2^2$.

Every point $\tilde{Q}$ of the unit circle ($\tilde{q}_1^2 + \tilde{q}_2^2 = 1$) is an improper face. Only one support line passes through such a point—the line tangent to the circle with the normal vector $P = \tilde{Q}$. Consequently, the normal cone of any face $\tilde{Q}$ is the ray $\{c\tilde{Q}, c > 0\}$.

Example 5. Let $\mathbf{D}$ be the set of points in the plane $\mathbf{R}_1^2$ with non-negative, integral coordinates. Then Γ is the first quadrant, $\Gamma = \{Q: Q \geqslant 0\}$. Its boundary $\partial\Gamma$ consists of one vertex and two edges:

$$\Gamma_1^{(0)} = \{0\} \ . \qquad \Gamma_1^{(1)} = \{Q: q_2 = 0 \leqslant q_1\} \ , \qquad \Gamma_2^{(1)} = \{Q: q_1 = 0 \leqslant q_2\} \ .$$

The normal cone for the vertex $\Gamma_1^{(0)}$ is

$$\mathbf{U}_1^{(0)} = \{P: \langle Q, P\rangle < 0 \quad \text{for all } Q \in \Gamma_j^{(1)} \backslash \Gamma_1^{(0)}\}$$

Choosing for Q the values $(1,0)$ and $(0,1)$, we get

$$\mathbf{U}_1^{(0)} = \{P: p_1 < 0, p_2 < 0\} \ .$$

The normal cones to the edges are

$$\mathbf{U}_1^{(1)} = \{P = (0, p_2): p_2 < 0\} \ , \qquad \mathbf{U}_2^{(1)} = \{P = (p_1, 0): p_1 < 0\} \ .$$

The cone $\mathbf{U} = \mathbf{U}_1^{(0)} \cup \mathbf{U}_1^{(1)} \cup \mathbf{U}_2^{(1)}$ is the third quadrant of the plane $\mathbf{R}_2^2$: $\mathbf{U} = \{P \leqslant 0\}$.

1.3. Degenerate Cases

It might happen that all the points of a set $\mathbf{D}$ are collinear. In this case, the set Γ lies entirely on this line and is a line-segment, a half-line, or the line itself. It has a single edge, $\Gamma_1^{(1)} = \Gamma$, and the number of vertices does not exceed two.

Example 6. $\mathbf{D} = \{Q_1, Q_2, Q_3\}$, $Q_1 = (2,0)$, $Q_2 = (0,2)$, $Q_3 = (1,1)$ (figure 4). Γ is the line-segment with vertices $\Gamma_1^{(0)} = Q_1$ and $\Gamma_2^{(0)} = Q_2$. There are three sets $\mathbf{D}_j^{(d)}$: $\mathbf{D}_1^{(0)} = Q_1$, $\mathbf{D}_2^{(0)} = Q_2$, and $\mathbf{D}_1^{(1)} = \mathbf{D}$. In constructing the normal cones, we

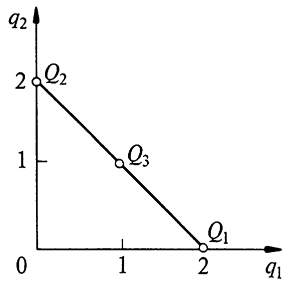

Fig. 4

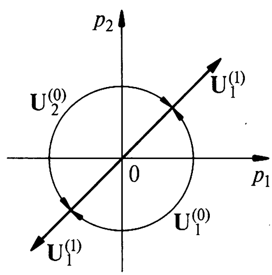

Fig. 5

find that there is one less inequality than in the non-degenerate case. Thus, we have (figure 5):

$$\mathbf{U}_1^{(1)} = \{P: \langle \Gamma_1^{(0)}, P\rangle = \langle \Gamma_2^{(0)}, P\rangle\} = \{P: p_1 = p_2\} \ .$$

This is the whole line $p_1 = p_2$—not a half-line as in the non-degenerate case.

$$\mathbf{U}_1^{(0)} = \{P: \langle \Gamma_2^{(0)}, P\rangle < \langle \Gamma_1^{(0)}, P\rangle\} = \{P: p_2 < p_1\} \ ;$$

$$\mathbf{U}_2^{(0)} = \{P: \langle \Gamma_1^{(0)}, P\rangle < \langle \Gamma_2^{(0)}, P\rangle\} = \{P: p_1 < p_2\} \ .$$

Here, $\mathbf{U} = \mathbf{U}_1^{(1)} \cup \mathbf{U}_1^{(0)} \cup \mathbf{U}_2^{(0)}$ is the entire $\mathbf{R}_2^2$ plane.

If the set $\mathbf{D}$ consists of a single point Q_1, then Γ degenerates to this single point. Thus, $\Gamma = \Gamma_1^{(0)} = Q_1 = \mathbf{D} = \mathbf{D}_1^{(0)}$. The normal cone $\mathbf{U}_1^{(0)}$ is the plane $\mathbf{R}_2^2$, as no conditions are placed on the vectors P.

Thus, every set $\mathbf{D} \subset \mathbf{R}_1^2$ has a corresponding set Γ which is the intersection of all the supporting half-planes of $\mathbf{D}$. Γ, in turn, has faces $\Gamma_j^{(d)}$, and for each $\Gamma_j^{(d)}$ there is a corresponding boundary subset $\mathbf{D}_j^{(d)} = \Gamma_j^{(d)} \cap \mathbf{D}$ and a normal cone $\mathbf{U}_j^{(d)}$. The order of these calculations may be written symbolically:

$$\mathbf{D} \to \Gamma \to \{\Gamma_j^{(d)}\} \to \{\mathbf{D}_j^{(d)}\} \to \{\mathbf{U}_j^{(d)}\} \ .$$

1.4. Polygons

We now consider the situation in which the set $\mathbf{D}$ consists of a finite number of points $Q_1, \ldots, Q_m$. Then, the convex hull Δ of $\mathbf{D}$, is closed and coincides with the set Γ (Theorem 1). Now, Γ is a polygon whose vertices are necessarily points of the set $\mathbf{D}$. Any remaining points of $\mathbf{D}$ will lie either on the edges of Γ or inside the polygon. It is also possible that an edge contains no points of $\mathbf{D}$ other than the vertices (see example 1).

The polygon Γ is the convex hull of a finite number of points and has a finite number of edges and vertices; these are the $\Gamma_j^{(d)}$. In principle, these could be arbitrarily indexed, but it is more convenient to proceed as follows. We choose a vertex and an orientation (clockwise, say). We designate the chosen vertex $\Gamma_1^{(0)}$, the next vertex in the chosen direction is $\Gamma_2^{(0)}$, then $\Gamma_3^{(0)}$, and so forth. Finally, we will come to some vertex $\Gamma_n^{(0)}$ which is followed by the original vertex $\Gamma_1^{(0)}$. Then we can number the edges by letting edge $\Gamma_j^{(1)}$ connect $\Gamma_j^{(0)}$ and $\Gamma_{j+1}^{(0)}$ for $j < n$, and connecting $\Gamma_n^{(0)}$ and $\Gamma_1^{(0)}$ with edge $\Gamma_n^{(1)}$. Thus, in example 1 we let $\Gamma_j^{(0)} = Q_j$ and numbered the edges as in figure 2.

The vertices $\Gamma_j^{(0)}$ play a special role in the polygon Γ. First, Γ is the convex hull of all its own vertices; second, each edge $\Gamma_j^{(1)}$ is the convex hull of its two vertices $\Gamma_j^{(0)}$ and $\Gamma_{j+1}^{(0)}$.

The cone $\mathbf{U}$ is, in this case, the entire $\mathbf{R}_2^2$ plane. The normal cones $\mathbf{U}_j^{(d)}$ to the faces $\Gamma_j^{(d)}$ are given by the formulas of (5). In example 3, the normal cones were defined quite simply, thanks to the simplicity of the set $\mathbf{D}$ (which consisted of just three points). Generally, the number of equations and inequalities defining the normal cones can be reduced by using just those points of the set $\mathbf{D}$ which are vertices $\Gamma_j^{(0)}$ of the polygon Γ. Then

$$\mathbf{U}_j^{(1)} = \left\{P\colon \begin{matrix} \langle \Gamma_k^{(0)}, P\rangle = \langle \Gamma_j^{(0)}, P\rangle; \Gamma_j^{(0)}, \Gamma_k^{(0)} \in \Gamma_j^{(1)}; \\ \langle \Gamma_i^{(0)}, P\rangle < \langle \Gamma_j^{(0)}, P\rangle, \Gamma_i^{(0)} \in \Gamma \backslash \Gamma_j^{(1)} \end{matrix}\right\},$$

$$\mathbf{U}_j^{(0)} = \{P\colon \langle \Gamma_i^{(0)}, P\rangle < \langle \Gamma_j^{(0)}, P\rangle, \Gamma_i^{(0)} \neq \Gamma_j^{(0)}\} .$$

Furthermore, in determining the normal cone $\mathbf{U}_j^{(1)}$ of an edge $\Gamma_j^{(1)}$ it is sufficient to use just one inequality, since the equation defines a line perpendicular to the edge $\Gamma_j^{(1)}$, and each inequality reduces to the same half-line, directed out of the polygon Γ.

In the determination of the normal cones $\mathbf{U}_j^{(0)}$ it is sufficient to use just two inequalities:

$$\mathbf{U}_j^{(0)} = \{P\colon \langle \Gamma_{j-1}^{(0)}, P\rangle < \langle \Gamma_j^{(0)}, P\rangle, \langle \Gamma_{j+1}^{(0)}, P\rangle < \langle \Gamma_j^{(0)}, P\rangle\} ,$$

that is, the normal cone of the vertex $\Gamma_j^{(0)}$ is bounded by the normals to the sides $\Gamma_{j-1}^{(1)}$ and $\Gamma_j^{(1)}$, which adjoin the vertex $\Gamma_j^{(0)}$. This follows from the fact that the inequalities defining the cone $\mathbf{U}_j^{(0)}$ can be written as $\langle \Gamma_i^{(0)} - \Gamma_j^{(0)}, P\rangle < 0$. All of these can be derived from the inequalities $\langle \Gamma_{j-1}^{(0)} - \Gamma_j^{(0)}, P\rangle < 0$ and $\langle \Gamma_{j+1}^{(0)} - \Gamma_j^{(0)}, P\rangle < 0$, since the vectors $\Gamma_i^{(0)} - \Gamma_j^{(0)}$ lie within the angle formed by

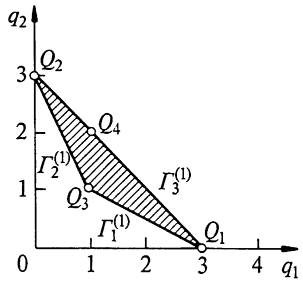

Fig. 6

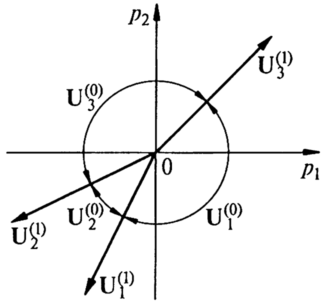

Fig. 7

the vectors $\Gamma_{j-1}^{(0)} - \Gamma_j^{(0)}$ and $\Gamma_{j+1}^{(0)} - \Gamma_j^{(0)}$. That is, $\Gamma_i^{(0)} - \Gamma_j^{(0)} = c_1(\Gamma_{j-1}^{(0)} - \Gamma_j^{(0)}) + c_2(\Gamma_{j+1}^{(0)} - \Gamma_j^{(0)})$, where $c_1, c_2 > 0$.

Example 7. $\mathbf{D} = \{Q_1, Q_2, Q_3, Q_4\}$, where $Q_1 = (3,0)$, $Q_2 = (0,3)$, $Q_3 = (1,1)$, $Q_4 = (1,2)$ (figure 6). The points Q_1, Q_2, Q_3, are the vertices of a triangle Γ. We let $\Gamma_1^{(0)} = Q_1$, $\Gamma_2^{(0)} = Q_3$, and $\Gamma_3^{(0)} = Q_2$. Then $\Gamma_1^{(1)}$ is the edge connecting Q_1 and Q_3, the edge $\Gamma_2^{(1)}$ connects Q_3 and Q_2, and the edge $\Gamma_3^{(1)}$ connects Q_2 and Q_1 (figure 6). Further the sets $\mathbf{D}_j^{(d)}$ are

$$\mathbf{D}_1^{(0)} = Q_1 \ , \qquad \mathbf{D}_2^{(0)} = Q_3 \ , \qquad \mathbf{D}_3^{(0)} = Q_2 \ ,$$

$$\mathbf{D}_1^{(1)} = \{Q_1, Q_3\} \ , \qquad \mathbf{D}_2^{(1)} = \{Q_3, Q_2\} \ , \qquad \mathbf{D}_3^{(1)} = \{Q_2, Q_4, Q_1\} \ .$$

We next find the normal cones $\mathbf{U}_j^{(d)}$. We have (figure 7):

$$\mathbf{U}_1^{(1)} = \{P: \langle \Gamma_1^{(0)}, P\rangle = \langle \Gamma_2^{(0)}, P\rangle > \langle \Gamma_3^{(0)}, P\rangle\} = \{P: p_2 = 2p_1 < 0\} \ ;$$

$$\mathbf{U}_2^{(1)} = \{P: \langle \Gamma_2^{(0)}, P\rangle = \langle \Gamma_3^{(0)}, P\rangle > \langle \Gamma_1^{(0)}, P\rangle\} = \{P: 2p_2 = p_1 < 0\} \ ;$$

$$\mathbf{U}_3^{(1)} = \{P: \langle \Gamma_3^{(0)}, P\rangle = \langle \Gamma_1^{(0)}, P\rangle > \langle \Gamma_2^{(0)}, P\rangle\} = \{P: p_1 = p_2 > 0\} \ ;$$

$$\mathbf{U}_1^{(0)} = \{P: \langle \Gamma_3^{(0)}, P\rangle < \langle \Gamma_1^{(0)}, P\rangle, \langle \Gamma_2^{(0)}, P\rangle < \langle \Gamma_1^{(0)}, P\rangle\}$$

$$= \{P: p_2 < p_1, p_2 < 2p_1\} \ ;$$

$$\mathbf{U}_2^{(0)} = \{P: \langle \Gamma_1^{(0)}, P\rangle < \langle \Gamma_2^{(0)}, P\rangle, \langle \Gamma_3^{(0)}, P\rangle < \langle \Gamma_2^{(0)}, P\rangle\}$$

$$= \{P: 2p_1 < p_2, 2p_2 < p_1\} \ ;$$

$$\mathbf{U}_3^{(0)} = \{P: \langle \Gamma_2^{(0)}, P\rangle < \langle \Gamma_3^{(0)}, P\rangle, \langle \Gamma_1^{(0)}, P\rangle < \langle \Gamma_3^{(0)}, P\rangle\}$$

$$= \{P: p_1 < 2p_2, p_2 > p_1\} \ .$$

1.5. Subsets of the Integral Lattice

The collection of all integral vectors $Q = (q_1, q_2)$ is called the *integral lattice* and denoted by $\mathbf{Z}^2$. We now consider the case in which $\mathbf{D} \subset \mathbf{Z}^2$ and indicate some properties of the sets Γ, $\mathbf{U}$, $\Gamma_j^{(d)}$, and $\mathbf{U}_j^{(d)}$.

1. Γ has no more than a countable set of faces $\Gamma_j^{(d)}$.

2. All the vertices $\Gamma_j^{(0)}$ are in $\mathbf{D}$, and two edges extend from each vertex.

3. Each edge $\Gamma_j^{(1)}$, which is a line segment, belongs to Δ, the convex hull of $\mathbf{D}$, and each is the convex hull of its own vertices.

4. The cones $\mathbf{U}_j^{(d)}$ of all the vertices and edges, which are line segments, can be found by formula (5).

We call the set

$$\mathbf{K} = \{P: P = \beta_1 P_1 + \beta_2 P_2; P_i \in \mathbf{D}, \beta_i \geqslant 0, i = 1, 2\} \ .$$

the *convex conic hull* of the set $\mathbf{D} \subset \mathbf{R}^2$. We call a convex cone *rational* if it is the convex conic hull of integral vectors. For example, the first quadrant $\{P \geqslant 0\}$ is a rational cone, since it is the convex conic hull of the vectors (1,0) and (0,1).

The normal cone $\mathbf{U}$ is the union of the cones $\mathbf{U}_j^{(d)}$, of which there may be a countable number.

Theorem 3. *If* $\mathbf{D} \subset \mathbf{Z}^2$, *then each rational cone* $\mathbf{K} \subset \mathbf{U}$ *intersects only a finite number of cones* $\mathbf{U}_j^{(d)}$.

We omit the proof.

If the set $\mathbf{D}$ lies in the first quadrant $\{Q \geqslant 0\}$ of $\mathbf{R}_1^2$, the normal cone $\mathbf{U}$ of $\mathbf{D}$ contains the third quadrant $\{P \leqslant 0\}$ of $\mathbf{R}_2^2$. Therefore, there will only be a finite number of cones $\mathbf{U}_j^{(d)}$ in the third quadrant of $\mathbf{R}_2^2$. Consequently, the set Γ has only a finite number of faces $\Gamma_j^{(d)}$ which lie on support lines $\mathbf{L}_P$ with $P \leqslant 0$. There are two methods of finding these faces.

First, it is possible to find all the faces $\Gamma_j^{(d)}$ and their cones $\mathbf{U}_j^{(d)}$, and then to select those cones which intersect the third quadrant of the plane $\mathbf{R}_2^2$.

Second, we can begin by finding the vertical and horizontal support lines of $\mathbf{D}$ with the normal vectors $P^* = (-1, 0)$ and $P_* = (0, -1)$, respectively. These

vectors span the rays $\mathbf{P}^* = (-c, 0)$ and $\mathbf{P}_* = (0, -c)$ (where $c > 0$), which in turn form the boundaries of the set $\{P < 0\}$. Let $q_{2*} = \min q_2$ for $Q \in \mathbf{D}$. Then the line $q_2 = q_{2*}$ will be the horizontal support line. Further, let $q_{1*} = \min q_1$ for $Q = (q_1, q_{2*}) \in \mathbf{D}$.

The point $Q_* = (q_{1*}, q_{2*})$ belongs to the set $\mathbf{D}$, and is the left-most point of this set on the horizontal support line $q_2 = q_{2*}$. We denote this point by $\Gamma_1^{(0)}$, and pass through it that slanting support line $\mathbf{L}_P$ $(P < 0)$ which contains one more point of $\mathbf{D}$. We then select the leftmost point of $\mathbf{D}$ on this line, denote it by $\Gamma_2^{(0)}$, and pass through this point a new support line which contains yet another point of $\mathbf{D}$ but does not contain $\Gamma_1^{(0)}$. We denote the leftmost point of $\mathbf{D}$ on this new line by $\Gamma_3^{(0)}$, and proceed as above. This procedure is continued until we come to a point $Q^* \in \mathbf{D}$ lying on the vertical support line below the remaining points of $\mathbf{D}$; that is, $Q^* = (q_1^*, q_2^*)$, where $q_1^* = \min q_1$ for all $Q \in \mathbf{D}$ and $q_2^* = \min q_2$ for all $Q = (q_1^*, q_2) \in \mathbf{D}$. As a vertex Q^* must be indexed: $\Gamma_k^{(0)} = Q^*$. A convex polygonal line $\hat{\Gamma}$ is contained between vertices $\Gamma_1^{(0)}$ and $\Gamma_k^{(0)}$; this open polygon consists of the vertices $\Gamma_j^{(0)}$ and the edges connecting them, $\Gamma_j^{(1)}$. All of these are vertices and edges of the polygon Γ, but not all the vertices and edges of the polygon Γ are in $\hat{\Gamma}$. In the situation of example 7 we have

$$Q_* = Q_1 = \Gamma_1^{(0)} , \qquad Q^* = Q_2 = \Gamma_3^{(0)} .$$

The open polygon $\hat{\Gamma}$ consists of the three vertices $\Gamma_1^{(0)}$, $\Gamma_2^{(0)}$, $\Gamma_3^{(0)}$ and the two edges, $\Gamma_1^{(1)}$ and $\Gamma_2^{(1)}$ (figure 6).

Thus we have

Theorem 4. *If $\mathbf{D} \subset \mathbf{Z}^2$ and lies in the first quadrant $\{Q \geqslant 0\}$, then the number of the boundary subsets $\mathbf{D}_P$, corresponding to vectors $P \leqslant 0$, is finite.*

The intersection Γ of the supporting half-planes of $\mathbf{D}$ is called *Newton's polygon*, the part $\hat{\Gamma}$ of its boundary is *Newton's open polygon*.

In earlier times the words "Newton's polygon" denoted the polygonal line $\hat{\Gamma}$, a part of the boundary of the convex hull Γ of a set $\mathbf{D}$. More recently, "Newton's polygon" has come to mean the convex hull itself—i.e., a two-dimensional set. We adhere to the more recent terminology, and use the term "Newton's open polygon" for $\hat{\Gamma}$.

Newton's open polygon is employed in finding the solutions, near the origin $x_1 = x_2 = 0$, of the equation

$$\sum f_{q_1 q_2} x_1^{q_1} x_2^{q_2} = 0 ,$$

where the exponents q_1 and q_2 are integral and non-negative (see below, §2). Newton himself [Newton, 1937] constructed only the lower right edge of the open polygon, the edge abutting the point Q_*, besides he used q_2 as abscissa and q_1 as ordinate. In Chapter 4 of Walker's book [Walker, 1950] Newton's polygon is described, though somewhat differently from our description (see also the review article by Chebotarev [1943]).

The aim of our geometrical constructions is the solution of the following **problem**: Given a set of two dimensional vectors, $\mathbf{D}$; for every vector $P \neq 0$ (or $P < 0$), choose a *boundary subset* $\mathbf{D}_P$ for which

$$\langle Q_i, P\rangle = \langle Q_j, P\rangle \ ; \qquad Q_i, Q_j \in \mathbf{D}_P \ ,$$

$$\langle Q_k, P\rangle < \langle Q_j, P\rangle \ ; \qquad Q_k \in \mathbf{D}\backslash\mathbf{D}_P \ .$$

It was shown above that solutions to this problem are the subsets $\mathbf{D}_j^{(d)}$, which lie on the faces $\Gamma_j^{(d)}$ of the Newton's open polygon $\hat{\Gamma}$ of $\mathbf{D}$. There are, however, other geometric methods of solving this problem. We will consider them below, assuming for simplicity that $\mathbf{D}$ consists of a finite number of points $Q_1, \ldots, Q_m$.

1.6. The Dual Open Polygon

Given a set $\mathbf{D}$, let us find its boundary subsets $\mathbf{D}_P$. To do this, in the s_1, s_2 plane we draw a line $\mathbf{Q}_j$ corresponding to each $Q_j \in \mathbf{D}$; the line $\mathbf{Q}_j$ is given by the equation

$$\langle Q_j, S\rangle = 1 \ .$$

We next take a fixed vector P and consider the line $\mathbf{P} = \{cP\}$, where c is an arbitrary real number. We designate by S_j the point of intersection of $\mathbf{P}$ and the line $\mathbf{Q}_j$, i.e.,

$$S_j = c_j P \ , \qquad \langle Q_j, S_j\rangle = 1 \ .$$

Therefore,

$$\langle Q_j, P\rangle = c_j^{-1}$$

and

$$\max\langle Q_j, P\rangle = \max c_j^{-1} = \begin{cases} (\min c_j)^{-1} \text{ for } c_j \geqslant 0 \ , & \text{if some } c_j \geqslant 0 \ , \\ (\min c_j)^{-1} \ , & \text{if all } c_j < 0 \ . \end{cases}$$

We divide the line $\mathbf{P}$ into two rays: $\mathbf{P}^{(+)} = \{cP, c \geqslant 0\}$ and $\mathbf{P}^{(-)} = \{cP, c < 0\}$. If the ray $\mathbf{P}^{(+)}$ intersects the lines $\mathbf{Q}_j$, then $\max(1/c_j)$ occurs at that point S_j on the ray which is nearest the origin $S_1 = S_2 = 0$. If the ray $\mathbf{P}^{(+)}$ does not intersect the lines $\mathbf{Q}_j$ then $\max(1/c_j)$ is attained at that point S_j of the ray $\mathbf{P}^{(-)}$ which is farthest from the origin. Thus, for every vector P, the set $\mathbf{D}_P$ consists of those vectors Q_j whose corresponding lines $\mathbf{Q}_j$ intersect the ray $\mathbf{P}^{(+)}$ nearest the origin. If no such lines Q_j exist, then the subset $\mathbf{D}_P$ is composed of those Q_j for which the lines $\mathbf{Q}_j$ intersect the ray $\mathbf{P}^{(-)}$ at the farthest point from the origin. Let us now consider all possible rays $\mathbf{P}^{(+)}$ and $\mathbf{P}^{(-)}$. Those points of intersection of the rays $\mathbf{P}^{(+)}$ with the lines $\mathbf{Q}_j$ which are nearest the origin make up a convex open polygon $\Omega^{(+)}$, while those intersections of the rays $\mathbf{P}^{(-)}$ with the lines $\mathbf{Q}_j$, which are furthest from the origin, form a convex open polygon $\Omega^{(-)}$. These open polygons are convex towards each other. A boundary subset $\mathbf{D}_P$ which consists of a single point Q_j corresponds to an edge of the open polygon $\Omega = \Omega^{(+)} \cup \Omega^{(-)}$. A vertex of Ω corresponds to the subset $\mathbf{D}_P$ consisting of all points Q_j for which the lines $\mathbf{Q}_j$ pass through this vertex.

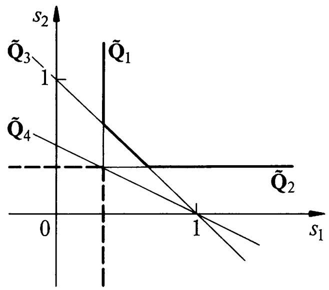

Fig. 8

The open polygons $\partial\Gamma$ and Ω are dual to each other, and lie in the dual planes $\mathbf{R}_1^2$ and $\mathbf{R}_2^2$. The s_1, s_2 plane considered above is $\mathbf{R}_2^2$, the plane for which we earlier used the coordinates p_1, p_2. (New coordinates were introduced for convenience.) There exists a one-to-one correspondence between the vertices and edges of $\partial\Gamma$ and Ω: each vertex of $\partial\Gamma$ corresponds to an edge of Ω, and each vertex of Ω corresponds to an edge of $\partial\Gamma$.

Example 8 (Compare example 7). Let $\mathbf{D} = \{Q_1, Q_2, Q_3, Q_4\}$, where $Q_1 = (3,0)$, $Q_2 = (0,3)$, $Q_3 = (1,1)$, and $Q_4 = (1,2)$. Then (see figure 8) we have the lines

$$\mathbf{Q}_1 = \{S: s_1 = \tfrac{1}{3}\} , \qquad \mathbf{Q}_2 = \{S: s_2 = \tfrac{1}{3}\} ,$$

$$\mathbf{Q}_3 = \{S: s_1 + s_2 = 1\} , \qquad \mathbf{Q}_4 = \{S: s_1 + 2s_2 = 1\} .$$

The open polygon $\Omega^{(-)}$ (the heavy lines in figure 8) consists of three sides and two vertices (compare figures 6 and 7); the open polygon $\Omega^{(+)}$ consists of two sides and one vertex (the heavy dashed lines).

1.7. Frommer's Method

The equation

$$\langle Q_j, P\rangle = q_{1j}p_1 + q_{2j}p_2 = r \tag{6}$$

is homogeneous in the variables p_1, p_2 and r. We can normalize any one of the three variables, and then equation (6) will define a line in the plane of the other two. Depending on which of p_1, p_2, or r is normalized (i.e., set equal to unity), we will have three possible geometric interpretations. In each case the vector $Q_j = (q_{1j}, q_{2j})$ corresponds to a line. We considered the case $r = 1$ in the previous section; here, we look at the case $p_1 = 1$. In this case, every vector $Q_j = (q_{1j}, q_{2j}) \in \mathbf{D}$ defines a line $\tilde{\mathbf{Q}}_j$ in the (p, r) plane according to the equation

$$q_{1j} + q_{2j}p = r . \tag{7}$$

For a fixed vector $P = (p_1, p_2)$ with $p_1 \neq 0$, we have $P = p_1(1, p_2/p_1)$; thus, $\langle Q_j, P\rangle = p_1(q_{1j} + q_{2j}p_2/p_1) = p_1 r_j$, where r_j is the ordinate corresponding to the abscissa $p = p_2/p_1$ on the line $\tilde{\mathbf{Q}}_j$. Therefore

$$\max_j \langle Q_j, P\rangle = \max_j p_1 r_j = \begin{cases} p_1 \max\limits_j r_j \ , & \text{if } p_1 > 0 \ , \\ p_1 \min\limits_j r_j \ , & \text{if } p_1 < 0 \ . \end{cases}$$

The boundary subset $\mathbf{D}_P$ consists of those points $Q \in \mathbf{D}$ for which $r = r^*(p) = \max_j r_j$ if $p_1 > 0$, or $r = r_*(p) = \min_j r_j$ if $p_1 < 0$. Geometrically, this means that for every value of the abscissa p in the (p, r) plane, we can find a largest $r^*(p)$ and a smallest $r_*(p)$ among the points of the lines $\tilde{\mathbf{Q}}_j$. Depending on the sign of p_1, we choose either the point $(p, r^*(p))$ or $(p, r_*(p))$ and select those lines $\tilde{\mathbf{Q}}_j$ which pass through the chosen point.

The elements of the set $\mathbf{D}$ which correspond to the selected lines determine the boundary subset $\mathbf{D}_P$. The collection of points $(p, r^*(p))$, for all p, is the open polygon $\Phi^{(+)}$; the collection of points $(p, r_*(p))$ is the open polygon $\Phi^{(-)}$. Each of the open polygons is convex, and convex towards the other. The edge of $\Phi = \Phi^{(+)} \cup \Phi^{(-)}$ corresponds to the single-point boundary subset $\mathbf{D}_P$, i.e., to the vertex of the open polygon $\partial\Gamma$. The vertex of the open polygon Φ corresponds to the multi-point boundary subset $\mathbf{D}_P$—i.e., to the edge of the open polygon $\partial\Gamma$.

Example 9 (see examples 7 and 8). For $\mathbf{D} = \{Q_1, Q_2, Q_3, Q_4\}$ where $Q_1 = (3,0)$, $Q_2 = (0,3)$, $Q_3 = (1,1)$, and $Q_4 = (1,2)$, we have the four lines (see fig. 9)

$$\tilde{\mathbf{Q}}_1 = \{3 = r\} \ , \qquad \tilde{\mathbf{Q}}_3 = \{1 + p = r\} \ ,$$

$$\tilde{\mathbf{Q}}_2 = \{3p = r\} \ , \qquad \tilde{\mathbf{Q}}_4 = \{1 + 2p = r\} \ .$$

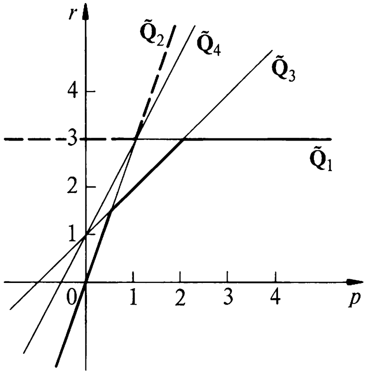

Fig. 9

The bold line represents the open polygon $\Phi^{(-)}$. Its three edges correspond to vertices Q_1, Q_2, and Q_3 of the Newton open polygon in figure 6. The two vertices of the open polygon $\Phi^{(-)}$ correspond to the edges $\Gamma_1^{(1)}$ and $\Gamma_2^{(1)}$ in figure 6. The dashed line in figure 9 is the open polygon $\Phi^{(+)}$, the two edges and single vertex of which correspond, respectively, to vertices Q_1 and Q_2 and to edge $\Gamma_3^{(1)}$ in figure 6.

This method of finding the boundary subset $\mathbf{D}_p$ with help of the open polygon Φ was first developed by Frommer [1928] for the study of differential equations. The development of this method has been carried out by Kukles [1958, 1964] and his students, as well as by Andreev [1970], Andreev and Gleiser [1972], and others. Note that the third possible normalization of equation (6), setting $p_2 = 1$, may be carried out in a similar fashion.

In comparison with the open polygon $\partial\Gamma$ and the dual polygon Ω, Frommer's open polygon Φ has the disadvantage that it is not symmetric relative to the coordinates q_1 and q_2 (or p_1, p_2). It is therefore difficult to construct a multi-dimensional generalization of Frommer's open polygon. Moreover, it is hard to determine the changes in Frommer's open polygon under power transformations (see below). From now on, therefore, we shall adopt the geometric interpretation of Newton. The use of Newton's polygon in solving differential equations was introduced by Briot and Bouquet [1856] and applied by Horn [1894], Dulac [1904], and others (see the surveys in § 20 of Dulac's monograph [Dulac, 1912] and in § 4, section 7 of Bieberbach's book [Bieberbach, 1953]). Apparently, Frommer did not read these works in the original, but instead read brief reviews written by Painlevé [1905]. Frommer therefore developed his own, less convenient method. One gets the impression that the many enthusiasts of "Frommer's method" never read the works of his predecessors. This is evident from their devotion to Frommer's open polygon, the absence of appropriate references, and the duplication of earlier results. For example, Dulac [1904] studied integral curves of zero and infinite order, but Frommer [1928] did not consider them at all, while Kukles [1958], devoted a series of works to the question, apparently not realizing it had already been studied (see also Andreev, 1970.) Beklemisheva [1971] suggests not normalizing p_1, p_2, or r in equation (6), considering the cone $\mathbf{M} = \{(p_1, p_2, r): q_{1j}p_1 + q_{2j}p_2 \leqslant r, j = 1, \ldots, m\}$ in real three dimensional space.

1.8. Affine Transformations

We next consider the linear coordinate transformation in the $\mathbf{R}_2^2$ plane

$$p_1' = \alpha_{11}p_1 + \alpha_{12}p_2 \,, \qquad p_2' = \alpha_{21}p_1 + \alpha_{22}p_2$$

with the non-singular matrix

$$\alpha = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix},$$

(i.e., $\det \alpha \neq 0$). In vector form, the transformation is written

$$P' = \alpha P \ . \tag{8}$$

The scalar product $\langle Q, P \rangle$ for $Q \in \mathbf{R}_1^2$ and $P \in \mathbf{R}_2^2$ will be invariant under the transformation if we effect the corresponding coordinate transformation in $\mathbf{R}_1^2$ given by

$$Q' = \alpha^{-1*} Q \ , \tag{9}$$

where α^* is the transpose of the matrix α. By the property of the scalar product, $\langle \beta Q, P \rangle = \langle Q, \beta^* P \rangle$, we have

$$\langle Q', P' \rangle = \langle \alpha^{-1*} Q, \alpha P \rangle = \langle Q, \alpha^{-1} \alpha P \rangle = \langle Q, P \rangle \ .$$

Thus, under the corresponding transformations (8) and (9) in the $\mathbf{R}_2^2$ and $\mathbf{R}_1^2$ planes, the scalar product of vectors in those planes is invariant. These planes are therefore *conjugate* or *dual* to each other.

The affine transformation (8) is a one-to-one mapping of the plane $\mathbf{R}_2^2$ onto itself. Under it, lines are mapped onto lines. For example, for a fixed vector Q_j, the line $\langle Q_j, P \rangle = c = \text{const}$ maps onto the line $\langle Q_j', P' \rangle = c$. Moreover, all linear inequalities are invariant. Let a set $\mathbf{D} \in \mathbf{R}_1^2$ be given, with corresponding sets Γ, $\mathbf{U}$, $\Gamma_j^{(d)}$, $\mathbf{D}_j^{(d)}$ and $\mathbf{U}_j^{(d)}$ constructed as described earlier. Let $\mathbf{D}'$ be the set obtained from $\mathbf{D}$ by applying transformation (9) and let the corresponding sets be Γ', $\mathbf{U}'$, $\Gamma_j^{(d)'}$, $\mathbf{D}_j^{(d)'}$, and $\mathbf{U}_j^{(d)'}$. Then

$$\Gamma' = \alpha^{*-1} \Gamma \ , \qquad \Gamma_j^{(d)'} = \alpha^{*-1} \Gamma_j^{(d)} \ , \qquad \mathbf{D}_j^{(d)'} = \alpha^{*-1} \mathbf{D}_j^{(d)} \ ;$$

$$\mathbf{U}' = \alpha \mathbf{U} \ , \qquad \mathbf{U}_j^{(d)'} = \alpha \mathbf{U}_j^{(d)} \ ,$$

That is, all of our earlier constructions are invariant under transformations (8) and (9).

Problem: Let $\Gamma_j^{(1)}$ be an edge (or line segment). Find a transformation of the form of (9) such that the edge $\Gamma_j^{(1)'} = \alpha^{*-1} \Gamma_j^{(1)}$ is parallel to one of the coordinate axes q_1' or q_2'.

Solution. Let the vector $P \in \mathbf{R}_2^2$ be perpendicular to $\Gamma_j^{(1)}$. If $P' = \alpha P$ lies along a coordinate axis, then the perpendicular $\Gamma_j^{(1)'}$ lies on the other coordinate axis. It is therefore sufficient to find a matrix α such that one of the coordinates of the vector αP vanishes. Such a matrix is, for instance,

$$\alpha = \begin{pmatrix} 1 & 0 \\ -p_2 & p_1 \end{pmatrix} \tag{10}$$

or

$$\alpha = \begin{pmatrix} 1 & 0 \\ -a & 1 \end{pmatrix} , \tag{11}$$

where $a = p_2/p_1$. Then the vector $P' = (p_1, 0)$ lies along the coordinate axis p_1',

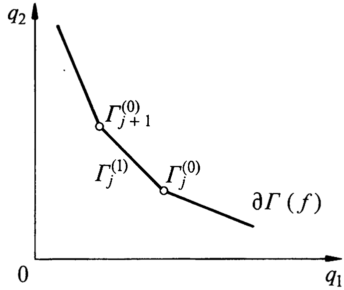

Fig. 10

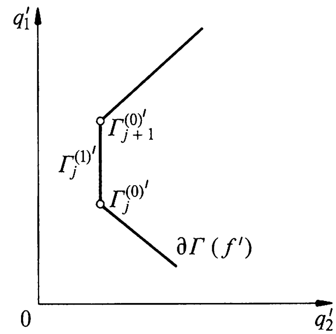

Fig. 11

and the edge $\Gamma_j^{(1)'}$ is parallel to the axis q_2' (figs. 10 and 11). We see that the inverse of matrix (11) is

$$\alpha^{-1} = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}.$$

In general, if

$$\beta = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}, \qquad \text{then} \quad \alpha\beta = \begin{pmatrix} 1 & 0 \\ b - a & 1 \end{pmatrix},$$

that is, under multiplication of such triangular matrices, the lower left-hand elements simply add.

Now consider a more complicated case of the previous problem. Let the set $\mathbf{D}$ lie in the first quadrant of the $\mathbf{R}_1^2$ plane, and let us require that $\mathbf{D}' = \alpha^{*-1}\mathbf{D}$ also lie in the first quadrant. We can assume that $P < 0$. Since $\mathbf{D}' = \alpha^{*-1}\mathbf{D}$, the requirement that the set $\mathbf{D}'$ lie in the first quadrant will automatically be satisfied if all the elements of the matrix α^{-1} are non-negative. But if $P < 0$, then $a = p_2/p_1$ is positive, and matrix (11) satisfies the requirement.

1.9. Unimodular Transformations

If the elements α_{ij} of the matrix α are integers, and if $\det\alpha = \pm 1$, then α is *unimodular*. The inverse of a unimodular matrix is also unimodular, and is given by

$$\alpha^{-1} = \begin{pmatrix} \alpha_{22} & -\alpha_{12} \\ -\alpha_{21} & \alpha_{11} \end{pmatrix} \det\alpha \ . \tag{12}$$

The main property of transformation (8) with a unimodular matrix α is that it is one-to-one on the integral lattice $\mathbf{Z}^2$.

Let us now solve the problem of the last section in the class of unimodular transformations (8), assuming that $\mathbf{D} \subset \mathbf{Z}^2$. In this case, all the vertices of the open polygon $\partial\Gamma$ are integral. Thus, given an edge $\Gamma_j^{(1)}$ with integral endpoints, we must find a unimodular transformation (8) under which its image, $\Gamma_j^{(1)'}$, is parallel to one of the coordinate axes.

Let $R = (r_1, r_2)$ be a single vector along the edge $\Gamma_j^{(1)}$, i.e., R is given by the differences between neighbouring integral points of the edge $\Gamma_j^{(1)}$. Then the vector

$$P = (p_1, p_2) = (-r_2, r_1) \tag{13}$$

will be perpendicular to $\Gamma_j^{(1)}$, where the integers r_1 and r_2 are relatively prime. We must find a unimodular matrix α such that one of the coordinates of the vector

$$\alpha P = (\alpha_{11}p_1 + \alpha_{12}p_2, \alpha_{21}p_1 + \alpha_{22}p_2) \tag{14}$$

vanishes. To simplify the following calculations, we assume that $P > 0$. We shall find a sequence of unimodular coordinate changes whose end result will be the desired transformation. For convenience, let $p_2 > p_1$. Division with remainder yields

$$p_2 = a_1 p_1 + p_2' \ ,$$

where $0 \leqslant p_2' < p_1$ and $a_1 > 0$, and p_2', a_1 are integers; $p_2' = 0$ only if $p_1 = 1$, since p_1 and p_2 are relatively prime. We make the transformation

$$p_1' = p_1 \ , \qquad p_2' = -a_1 p_1 + p_2$$

with the unimodular matrix

$$\alpha_1 = \begin{pmatrix} 1 & 0 \\ -a_1 & 1 \end{pmatrix} .$$

Under this transformation, the vector P transforms to a vector $P' = (p_1', p_2')$. If $p_1 = 1$, then $p_2' = 0$, and we have found the desired transformation. Otherwise, if $p_1 > 1$ then $p_2' > 0$ and the numbers p_1' and p_2' are relatively prime; we have not yet found the desired transformation, though the coordinates of P' are in a sense "smaller" than those of P. Let us continue the process of diminishing the coordinates. We divide p_1' by p_2',

$$p_1' = a_2 p_2' + p_1'' \ ,$$

where a_2 and p_1'' are integers, $0 \leqslant p_1'' < p_2'$, $a_2 > 0$, and p_1'' is the remainder. We have $p_2'' = 0$ only when $p_2' = 1$. We apply the unimodular transformation

$$p_1'' = p_1' - a_2 p_2' \ ,$$

$$p_2'' = p_2'$$

with matrix

$$\alpha_2 = \begin{pmatrix} 1 & -a_2 \\ 0 & 1 \end{pmatrix} .$$

Applying this transformation to P' yields a vector $P'' = (p_1'', p_2'') \geqslant 0$. If $p_2' = 1$, then $p_1'' = 0$ and the desired transformation will be the product of the two transformations:

$$\alpha = \alpha_2 \alpha_1 \ .$$

If $p_2' \neq 1$, then $p_1'' > 0$ and p_1'' and p_2'' are relatively prime. Again, the coordinates of the vector P'' are "less" than those of P'. We need to continue the process. Suppose that the k-th step yields a vector $P^{(k)} = (p_1^{(k)}, p_2^{(k)}) \geqslant 0$. If the larger of the coordinates equals one, the other must be zero. Then the desired matrix α is

$$\alpha = \alpha_k \alpha_{k-1} \ldots \alpha_1 \ ,$$

and the vector αP is simply one of the unit coordinate vectors. If, on the other hand, the larger of the two coordinates of $P^{(k)}$ is not unity, then the other is non-zero, since they are relatively prime. Dividing the larger coordinate by the smaller, we make the unimodular transformation $P^{(k+1)} = \alpha_{k+1} P^{(k)}$, where the vector $P^{(k+1)}$ is "smaller" than $P^{(k)}$. Continuing this process, at some n-th step we obtain the unit vector $P^{(n)}$. Then

$$\alpha = \alpha_n \alpha_{n-1} \ldots \alpha_2 \alpha_1 \ , \tag{15}$$

and the vector $P^{(n)} = \alpha P$ is a unit vector; our problem is solved.

Note that each matrix α_k is triangular, with ones along the diagonal and with the position of the third element dependent on the parity of k:

$$\begin{pmatrix} 1 & 0 \\ -a_k & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & -a_k \\ 0 & 1 \end{pmatrix} . \tag{16}$$

Example 10. Let $p_1 = 5, p_2 = 17$. Then $p_2 = 3p_1 + 2$, i.e., $a_1 = 3, p_1' = p_1 = 5$, $p_2' = 2$. Further, $p_1' = 2p_2' + 1$, so that $a_2 = 2$, $p_1'' = 1$, $p_2'' = p_2' = 2$. Finally, $p_2'' = 2p_1''$, and $a_3 = 2$, $p_1''' = 1$, and $p_2''' = 0$. Thus,

$$\alpha_1 = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}, \quad \alpha_3 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} .$$

As a result,

$$\alpha = \alpha_3\alpha_2\alpha_1 = \begin{pmatrix} 7 & -2 \\ -17 & 5 \end{pmatrix},$$

$$\alpha P = P''' = (1,0) .$$

Note that by (15)

$$\alpha^{-1} = \alpha_1^{-1}\alpha_2^{-1}\ldots\alpha_{n-1}^{-1}\alpha_n^{-1} ,$$

and that the matrices α_k^{-1} have the form

$$\begin{pmatrix} 1 & 0 \\ a_k & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & a_k \\ 0 & 1 \end{pmatrix},$$

each α_k^{-1} has all non-negative elements. As a result, the matrix α^{-1} has non-negative elements. It is essential here that $P > 0$.

In this construction we have used the algorithm of successive division with remainder, i.e., the Euclidean algorithm. In fact, we have sought a unimodular matrix α such that, for two relatively prime numbers p_1 and p_2, the vector (14) should be a unit vector. That is,

$$\begin{aligned} \alpha_{11}p_1 + \alpha_{12}p_2 &= 1 , \\ \alpha_{21}p_1 + \alpha_{22}p_2 &= 0 , \\ \det\alpha = \alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} &= \pm 1 . \end{aligned} \tag{17}$$

The second of these equations implies that

$$\alpha_{21} = -p_2 , \qquad \alpha_{22} = p_1 . \tag{18}$$

Then the first and third equations of (17) are equivalent. The solution of the first for integers $\alpha_{11} > 0$ and $\alpha_{12} < 0$ is simply a matter of applications of the Euclidean algorithm as seen above.

This process can also be interpreted as the expansion of the ratio $\gamma = p_1/p_2$ into a continued fraction

$$\gamma = \frac{p_1}{p_2} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}} .$$

Here, a_0 is the integral part of the number γ, a_1 is the integral part of $(\gamma - a_0)^{-1}$,

a_2 is the integral part of $((\gamma - a_0)^{-1} - a_1)^{-1}$, etc. Thus, in order to express γ as a continued fraction, we must separate γ into its integral part a_0 and a remainder, then separate the inverse of the remainder into an integral part and a new remainder, invert that remainder and find its integral part, and so forth. The theory of continued fractions (see Khinchin, 1961) tells us that the continued-fraction representation of a rational number is finite, and that the number

$$\rho = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_{n-1}}}}$$

is such that

$$\gamma - \rho = \frac{\sigma}{\tilde{q}_{n-1}\tilde{q}_n}, \tag{19}$$

where $\tilde{q}_{n-1}$ and $\tilde{q}_n$ are the denominators of ρ and γ respectively, and $\sigma = \pm 1$. Thus, $\tilde{q}_n = p_2$. If we write $\rho = \tilde{p}_{n-1}/\tilde{q}_{n-1}$, then, in agreement with (19),

$$p_1\tilde{q}_{n-1} - p_2\tilde{p}_{n-1} = \sigma = \pm 1 .$$

To satisfy equation (17), we need to set

$$\alpha_{11} = \tilde{q}_{n-1}\sigma , \qquad \alpha_{12} = -\tilde{p}_{n-1}\sigma . \tag{20}$$

Example 11 (a continuation of example 10). Let $p_1 = 5$, $p_2 = 17$. Then

$$\gamma = \frac{5}{17} = 0 + \cfrac{1}{3 + \cfrac{1}{2 + \frac{1}{2}}},$$

$$\rho = 0 + \frac{1}{3 + \frac{1}{2}} = \frac{2}{7} = \tilde{p}_{n-1}/\tilde{q}_{n-1} .$$

Since

$$\gamma - \rho = \frac{5}{17} - \frac{2}{7} = \frac{1}{119},$$

$\sigma = 1$. In agreement with (20) $\alpha_{11} = 7$, $\alpha_{12} = -2$. This, along with (18), gives us the matrix

$$\alpha = \begin{pmatrix} 7 & -2 \\ -17 & 5 \end{pmatrix} .$$

as we found earlier.

Remark. If $\alpha P = (1,0)$, then $\tilde{\alpha} P = (1,0)$ for

$$\tilde{\alpha} = \alpha + k\begin{pmatrix} \alpha_{21} & \alpha_{22} \\ 0 & 0 \end{pmatrix},$$

where k is an arbitrary integer.

We have described here solutions only to problems with positive P. If $P < 0$, we simply find the matrix α which solves the problem for $-P$. But if $p_1 p_2 < 0$, we find the unimodular matrix α for the vector $(|p_1|, |p_2|)$; then the matrix

$$\begin{pmatrix} \alpha_{11} \operatorname{sgn} p_1 & \alpha_{12} \operatorname{sgn} p_2 \\ \alpha_{21} \operatorname{sgn} p_1 & \alpha_{22} \operatorname{sgn} p_2 \end{pmatrix}$$

will be the unimodular matrix that causes one of the coordinates of P to vanish.

Exercise 1. Let $p_1 = 5$, $p_2 = 7$. Find a unimodular matrix α such that one of the coordinates of αP vanishes.

§ 2. Zeros of an Analytic Function

2.1. A Simple Point

Let the function $f(x_1, x_2)$ be analytic at the point $x_1 = x_2 = 0$. Then f may be expanded in a Taylor series

$$f(x_1, x_2) = \sum_{q_1, q_2 \geqslant 0} f_{q_1 q_2} x_1^{q_1} x_2^{q_2} \ , \tag{1}$$

which converges absolutely in some neighborhood $\mathcal{U} = \{x_1, x_2 \colon |x_1|, |x_2| < \varepsilon\}$ of the origin. In expression (1), q_1 and q_2 are non-negative integers and the coefficients $f_{q_1 q_2}$ are real or complex constants. The values of x_1 and x_2 may likewise be either real or complex. The presentation given here will be for the case of a real-valued function f of two real variables x_1 and x_2; however, the entire discussion applies equally well to the complex case.

Let us write the first few terms of the Taylor series for f:

$$f = f_{00} + f_{10}x_1 + f_{01}x_2 + f_{20}x_1^2 + f_{11}x_1x_2 + f_{02}x_2^2 + \cdots$$

Now let $f_{00} = 0$ (so that $f(0,0) = 0$), and consider the equation

$$f(x_1, x_2) = 0 \ . \tag{2}$$

The collection of those solutions of equation (2) which lie in the neighborhood $\mathcal{U}$ forms an *analytic set* (see Fuks, 1962)—specifically, a plane *analytic curve* passing through the origin. This section (§ 2) aims to study the location of this curve in the plane $\mathbf{R}_0^2$ in a sufficiently small neighborhood of the origin.

If $|f_{10}| + |f_{01}| \neq 0$, then $X = 0$ (the origin) is called a *simple* point for the curve defined by equation (2); otherwise, the origin is called a *critical* point. We begin with the solution of equation (2) in the neighborhood of a simple point, assuming that the coefficient f_{01} is non-zero.

Theorem 1: *Let a function f be analytic at the origin, with $f(0,0) = 0$ and $f_{01} \neq 0$. Then equation* (2) *has an analytic solution*

$$x_2 = b(x_1) = \sum_{k=1}^{\infty} b_k x_1^k \ . \tag{3}$$

Proof. We apply the method of undetermined coefficients. After substituting the solution (3) into the expansion (1), equation (2) must be an identity in x_1:

$$\sum_{q_1, q_2} f_{q_1 q_2} x_1^{q_1} \left(\sum_{k=1}^{\infty} b_k x_1^k \right)^{q_2} \equiv 0 \ .$$

Multiplying out and collecting terms, we obtain

$$\sum_{l=1}^{\infty} \left(\sum f_{q_1 q_2} b_{k_1} \dots b_{k_s} \right) x_1^l \equiv 0 \ , \tag{4}$$

where the indices of the internal sum are restricted by

$$q_1 + k_1 + \dots + k_s = l \ . \tag{5}$$

Identity (4) is satisfied if the coefficient of x_1^l vanishes for all $l = 1, 2, \dots$ that is, if each of the internal sums vanishes. In every such sum, only the term $f_{01} b_l$ contains the coefficient b_k for $k = l$; in all other terms, the indices of the b_k are strictly less than l, since in (5) $q_1 \geqslant 0$ and $k_j \geqslant 1$. We denote the sum of these remaining terms by c_l. Then identity (4) is equivalent to an infinite system of equations

$$f_{01} b_l + c_l(b_1, \dots, b_{l-1}) = 0 \ , \qquad l = 1, 2, \dots$$

From this system, we determine that $b_l = -c_l/f_{01}$ for each successive l.

By means of the method of majorants, we can show that series (3) converges for sufficiently small $|x_1|$. This is the Cauchy implicit function theorem (see Goursat, 1933, § 184). We will not, however, prove the convergence of series (3) here. In general, we will give results on the convergence and divergence of the series without proof. □

The solution $x_2 = b(x_1)$ of equation (2) is a simple curve in the (x_1, x_2) plane, passing through the origin.

Several methods may be used to calculate expansion (3). For example, we can express the polynomials c_l as functions of the coefficients b_j and $f_{q_1 q_2}$. Alternatively, we can employ successive approximations to find the series (3). To do so, we make the coordinate change

$$x_2 = b_1 x_1 + y_1$$

in the original equation (2). Then equation (2) takes the form

$$f_1(x_1, y_1) = 0 \ ;$$

where we must seek a solution

$$y_1 = \sum_{k=2}^{\infty} b_k x_1^k \ . \tag{6}$$

In subsequent approximations, we have

$$y_k = b_{k+1} x_1^{k+1} + y_{k+1} \ .$$

Example 1. Let $f = x_1 + x_2 + x_1 x_2$. In the notation of (1), $f_{10} = f_{01} = f_{11} = 1, f_{20} = f_{02} = 0$ and $f_{q_1 q_2} = 0$ if $q_1 + q_2 > 2$. We substitute $x_2 = b_1 x_1 + y_1$ into the expression for f and obtain the equation

$$x_1 + b_1 x_1 + y_1 + x_1(b_1 x_1 + y_1) = 0 \ .$$

It is evident from expression (6) that y_1 is of greater than linear order in x_1. Hence we can isolate all the terms of less than second order and obtain the equation $x_1 + b_1 x_1 = 0$; that is, $b_1 = -1$. We thus have the equation $y_1 - x_1^2 + y_1 x_1 = 0$. For a second approximation, $y_1 = b_2 x_1^2 + y_2$, we have

$$f = b_2 x_1^2 + y_2 - x_1^2 + b_2 x_1^3 + x_1 y_2 = 0 \ .$$

From the terms of second order in x_1, we obtain the equation $b_2 x_1^2 - x_1^2 = 0$; hence $b_2 = 1$, and $f = y_2 + x_1^3 + x_1 y_2$. Proceeding similarly with the third approximation $y_2 = b_3 x_1^3 + y_3$, we find that $f = b_3 x_1^3 + y_3 + x_1^3 + b_3 x_1^4 + x_1 y_3 = 0$. The terms of least order (here, third) give $b_3 x_1^3 + x_1^3 = 0$; that is, $b_3 = -1$. Proceeding further will give us the series $b = \sum_{k=1}^{\infty} (-x_1)^k$. On the other hand the equation $x_1 + x_2 + x_1 x_2 = 0$ is easily solved as a linear equation; its solution is $x_2 = -x_1/(1 + x_1)$. Writing this solution as a Taylor series about zero yields the same series, b, that we found above.

Before studying the behavior of analytic curves in the neighborhoods of critical points, let us examine some of the local properties of curves in the plane, as well as the asymptotic behavior of an analytic function on such curves.

2.2. The Order of Magnitude of a Function

Let τ be a real parameter. We will consider the behavior of a function of τ, defined for $\tau > \tau_0$, as $\tau \to +\infty$. By $o(1)$ we will denote any function of τ which approaches zero as $\tau \to +\infty$. If

$$\varphi(\tau) = b\tau^p (1 + o(1)) \ , \tag{7}$$

where b and p are real numbers, then as $\tau \to \infty$ we have

$$\varphi(\tau) \to \begin{cases} 0 \ , & \text{if } p < 0 \ , \\ b \ , & \text{if } p = 0 \ , \\ \infty \ , & p > 0 \ . \end{cases}$$

Let

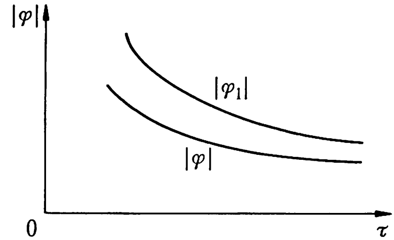

Fig. 12

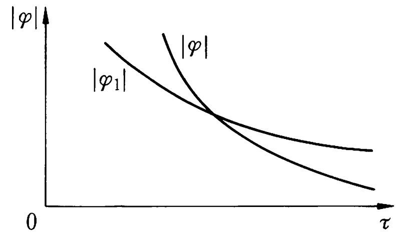

Fig. 13

$$\varphi_1(\tau) = b_1 \tau^{p_1}(1 + o(1)) \ .$$

If $p_1 > p$, then for sufficiently large τ we will have $|\varphi_1| > |\varphi|$, and the graphs of the two functions will look like those in figure 12. If $p_1 = p$ and $|b_1| > |b|$, then $|\varphi_1| > |\varphi|$ and the graphs will be similar. As an example, consider $\varphi = 1000\,\tau^{-3}$ and $\varphi_1 = \tau^{-2}$; this is illustrated in figure 13. Note that for very large τ, these curves behave just like those in figure 12. Note also that when $p_1 > p$,

$$\varphi + \varphi_1 = b_1 \tau^{p_1}(1 + o(1)) \ ,$$

$$\varphi\varphi_1 = bb_1 \tau^{p+p_1}(1 + o(1)) \ .$$

Finally, note that under the parameter change

$$\tau = c\tau^{*\alpha}(1 + o(1)) \ , \tag{8}$$

where $c > 0$ and $\alpha > 0$, function (7) becomes a function $\varphi^*(\tau^*) = \varphi(\tau) = bc^p\tau^{*p\alpha}(1 + o(1)) = b^*\tau^{*p^*}(1 + o(1))$, having the same form, but with a new exponent $p^* = \alpha p$ and with a new coefficient $b^* = bc^p$.

2.3. The Order of Magnitude of a Plane Curve

We now consider a curve $\mathscr{F}$ in the x_1, x_2 plane, defined by the pair of parametric equations

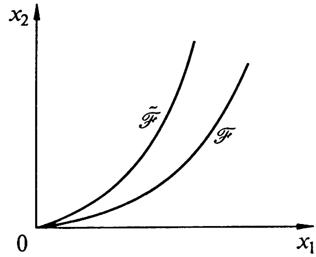

Fig. 14

$$\mathscr{F}: \begin{cases} x_1 = b_1 \tau^{p_1}(1 + o(1)) \ , \\ x_2 = b_2 \tau^{p_2}(1 + o(1)) \ . \end{cases} \tag{9}$$

We call any curve defined by such a parametrization a *curve of class* $\mathscr{W}$. We will examine the behavior of such curves in the neighborhood of the origin $X = 0$; then $p_1 < 0$ and $p_2 < 0$. We also assume for convenience in the following discussion that $b_1 > 0$ and $b_2 > 0$. Graphs of curves of class $\mathscr{W}$ are shown in figure 14.

We call the vector $P = (p_1, p_2)$ the (vector) *order of the curve* $\mathscr{F}$. If we replace τ with a new parameter τ^*, according to formula (8), the equations defining $\mathscr{F}$ become

$$x_1 = b_1 c^{p_1} \tau^{*p_1 \alpha}(1 + o(1)) = b_1^* \tau^{*p_1^*}(1 + o(1)) \ ,$$

$$x_2 = b_2 c^{p_2} \tau^{*p_2 \alpha}(1 + o(1)) = b_2^* \tau^{*p_2^*}(1 + o(1)) \ .$$

That is, $\mathscr{F}$ is a curve of class $\mathscr{W}$ with respect to τ^*, and its vector order with respect to the new parameter is

$$P^* = (p_1^*, p_2^*) = (\alpha p_1, \alpha p_2) = \alpha P \ .$$

Thus, to each curve $\mathscr{F}$ defined with respect to different parametrizations, given by (8), there correspond vector orders P^* which lie on the ray $\mathbf{P} = \{P^*\colon P^* = \alpha P, \alpha > 0\}$.

For example, the graph of the curve

$$\mathscr{F}: \begin{cases} x_1 = \tau^{-1} \ , \\ x_2 = \tau^{-3} \end{cases} \tag{10}$$

appears in figure 14; the curve's order vector $P = (-1, -3)$ and the ray $\mathbf{P}$ are represented in figure 15. Curve (10) is given by the single equation $x_2 = x_1^3$. In general, if we define $\mathscr{F}$ using the variable x_1^{-1} as the parameter τ, then the first coordinate of the vector order P is $p_1 = -1$.

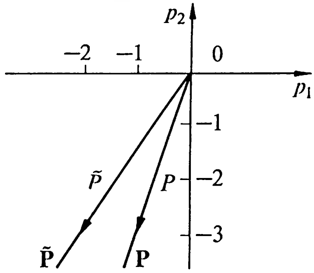

Fig. 15

Now suppose that, in addition to the curve (9), we have the curve

$$\tilde{\mathscr{F}}: \begin{cases} x_1 = \tilde{b}_1 \tilde{\tau}^{\tilde{p}_1}(1 + o(1)) \ , \\ x_2 = \tilde{b}_2 \tilde{\tau}^{\tilde{p}_2}(1 + o(1)) \ . \end{cases}$$

Let us consider the curves $\mathscr{F}$ and $\tilde{\mathscr{F}}$ in the x_1, x_2 plane (figure 14) as well as the corresponding order rays $\mathbf{P}$ and $\tilde{\mathbf{P}}$ in the p_1, p_2 plane (figure 15). We note that, in a sufficiently small neighborhood of the origin, $\mathscr{F}$ lies to the right of $\tilde{\mathscr{F}}$ as long as $\mathbf{P}$ lies to the right of $\tilde{\mathbf{P}}$. This is easy to show if we choose x_1^{-1} as the parameter. For example, let $\mathscr{F}$ be given by formula (10), and let $\tilde{\mathscr{F}}$ be given by

$$\tilde{\mathscr{F}}: \begin{cases} x_1 = \tau_1^{-2} \ , \\ x_2 = \tau_1^{-3} \ , \end{cases}$$

i.e. $x_2 = x_1^{3/2}$; then $\tilde{P} = (-2, -3)$ (see figures 14 and 15). There is thus a definite connection between the geometry of curves in a small neighborhood of the origin in the x_1, x_2 plane and the geometry of their vector orders in the p_1, p_2 plane.

Next we turn to general curves of the form

$$x_1 = b_1 \tau^{p_1} \ ,$$

$$x_2 = b_2 \tau^{p_2} \ .$$

We say that such curves are of the *class* $\mathscr{W}_0$. Every curve (9) of the class $\mathscr{W}$ can be confined between two curves of the class $\mathscr{W}_0$ in a sufficiently small neighborhood of the origin.

Let us consider two such curves of class $\mathscr{W}_0$:

$$\mathscr{F}^*: \{x_1 = b_1 \tau^{p_1}, x_2 = (b_2 + \varepsilon)\tau^{p_2}\} \ ,$$

$$\mathscr{F}_*: \{x_1 = b_1 \tau^{p_1}, x_2 = (b_2 - \varepsilon)\tau^{p_2}\} \ .$$

Comparing them with curve (9), we see that the new curves have exactly the same

order $P = (p_1, p_2)$ and differ only in the coefficients b_2, $b_2 + \varepsilon$, and $b_2 - \varepsilon$ of the x_2 coordinate. The result of this change of coefficients is that the curves $\mathscr{F}^*$ and $\mathscr{F}_*$ lie, respectively, just above and just below the curve $\mathscr{F}$.

Exercise 1. Explain the behavior of points on curve (9) as $\tau \to \infty$, particularly its dependence on the vector order $P = (p_1, p_2)$ of the curve (consider those cases in which P lies in different quadrants or along coordinate axes of the p_1, p_2 plane).

2.4. Asymptotics of a Function on a Curve

We consider the asymptotic behavior of a function $f(x_1, x_2)$ on a curve of form (9) as $\tau \to \infty$. We begin with the case in which f is a monomial i.e., $f = x_1^{q_1} x_2^{q_2}$. Then along a curve $\mathscr{F}$,

$$f_{\mathscr{F}} = b_1^{q_1} b_2^{q_2} \tau^{p_1 q_1 + p_2 q_2}(1 + o(1)) \ .$$

The order of this function with respect to τ is

$$q_1 p_1 + q_2 p_2 = \langle Q, P \rangle \ .$$

Next, consider the finite sum

$$f = \sum_{j=1}^{m} a_j x_1^{q_{1j}} x_2^{q_{2j}}$$

with real coefficients a_j. Then

$$f_{\mathscr{F}} = \sum_{j=1}^{m} a_j b_1^{q_{1j}} b_2^{q_{2j}} \tau^{p_1 q_{1j} + p_2 q_{2j}}(1 + o(1)) \ . \tag{11}$$

The exponents of τ in the terms of this sum will be different for different j; if we write $Q_j = (q_{1j}, q_{2j})$, then the exponents can be written as scalar products, $\langle Q_j, P \rangle$.

Since the terms of sum (11) with larger exponents will have larger magnitudes as $\tau \to \infty$, we can select the indices j of those terms with the largest exponent: $r = \max_j \langle Q_j, P \rangle$, P fixed. Let this maximum be reached for terms with indices $j = j_1, \ldots, j_s$. Then

$$\begin{cases} \langle Q_j, P \rangle = r & \text{for } j = j_1, \ldots, j_s \ ; \\ \langle Q_j, P \rangle < r & \text{for } j \neq j_1, \ldots, j_s \ . \end{cases} \tag{12}$$

Let us select all those terms for which $\langle Q_j, P \rangle = r$, and create a new function

$$\hat{f}(x_1, x_2) = \sum_{k=1}^{s} a_{j_k} x_1^{q_{1j_k}} x_2^{q_{2j_k}} \ .$$

Then on the curve $\mathscr{F}$

$$f_{\mathscr{F}} = \hat{f}_{\mathscr{F}} + o(1)\tau^r = \hat{f}(b_1, b_2)\tau^r + \tau^r o(1) \ . \tag{13}$$

The function $\hat{f}$ is called the first approximation to f for the vector order P or the *truncation of the function f for the* (*vector*) *order P*. Clearly, such a construction uses only the order P of the curve $\mathscr{F}$. Since the truncation $\hat{f}$ is determined by the leading term of f on the curve $\mathscr{F}$, we must know how to find the first approximation to f, considering the fact that different orders P will correspond to different truncations $\hat{f}$ of the function f.

2.5. Finding the Truncation

Let $\mathbf{D} = \{Q_1, \ldots, Q_m\}$ be a set of vector exponents for the sum f. For a fixed vector P, the truncation $\hat{f}$ will consist of terms $a_j x_1^{q_{1j}} x_2^{q_{2j}}$ whose vector exponent $Q_j = (q_{1j}, q_{2j})$ satisfies conditions (12). These conditions determine a boundary subset $\mathbf{D}_P$ of the set $\mathbf{D}$. Therefore,

$$\hat{f} = \sum_{Q_j \in \mathbf{D}_P} a_j x_1^{q_{1j}} x_2^{q_{2j}} \ . \tag{14}$$

The problem of choosing the boundary subsets $\mathbf{D}_P$ was discussed in § 1, where we presented the following geometrical method: from the set $\mathbf{D}$ we construct the Newton polygon Γ, whose boundary $\partial\Gamma$ consists of vertices $\Gamma_j^{(0)}$ and edges $\Gamma_j^{(1)}$; these correspond, respectively, to the normal cones $\mathbf{U}_j^{(0)}$ and $\mathbf{U}_j^{(1)}$. The boundary subset $\mathbf{D}_P$ is the set $\mathbf{D}_j^{(d)} = \mathbf{D} \cap \Gamma_j^{(d)}$, where $P \in \mathbf{U}_j^{(d)}$. For a fixed $\mathbf{U}_j^{(d)}$, to all $P \in \mathbf{U}_j^{(d)}$ will correspond a unique boundary subset $\mathbf{D}_P = \mathbf{D}_j^{(d)}$, and thus a unique truncation $\hat{f} = \hat{f}_j^{(d)}$ of the polynomial f. Hereafter, the normal cone $\mathbf{U}_j^{(d)}$ will also be called the *cone of the truncation* $\hat{f} = \hat{f}_j^{(d)}$.

Example 2 (see examples 1 and 3 of § 1). Let $f = x_1 + x_2 + x_1 x_2$. Here $Q_1 = (1,0)$, $Q_2 = (0,1)$, and $Q_3 = (1,1)$; $\mathbf{D} = \{Q_1, Q_2, Q_3\}$. The Newton polygon Γ and the division of the $\mathbf{R}_2^2$ plane into the normal cones $\mathbf{U}_j^{(d)}$ are pictured in figures 2 and 3, respectively. If $P = (-1, -1)$ then $P \in \mathbf{U}_1^{(1)}$, and $\mathbf{D}_P = \{Q_1, Q_2\}$. Thus, $\hat{f} = x_1 + x_2$.

A general scheme for constructing truncations and their cones may be described as follows. From the sum (1) we can construct a corresponding set $\mathbf{D}$ consisting of those points $Q = (q_1, q_2)$ which are the vector exponents of the terms in the sum f with non-zero coefficients, $f_{q_1 q_2} \neq 0$. The set $\mathbf{D}$ is called the *support* of sum (1), and is sometimes denoted by $\operatorname{supp} f$. From the set $\mathbf{D}$ we can construct the Newton polygon $\Gamma = \Gamma(f)$ and find its vertices and edges $\Gamma_j^{(d)}$. For each of the $\Gamma_j^{(d)}$, there is a corresponding boundary subset $\mathbf{D}_j^{(d)} = \Gamma_j^{(d)} \cap \mathbf{D}$. This subset $\mathbf{D}_j^{(d)}$ gives us the truncation $\hat{f}_j^{(d)}$ by formula (14). Finally, for each truncation we can construct a cone $\mathbf{U}_j^{(d)}$; this is the set of all vector orders P for which $\hat{f}_j^{(d)}$ is the first approximation to f. The described sequence of operations may be visually represented as the following scheme:

$$\begin{array}{ccccccccc} f & & & & & & \{\hat{f}_j^{(d)}\} & & \\ \downarrow & & & & & & \uparrow & & \\ \mathbf{D} & \rightarrow & \Gamma & \rightarrow & \{\Gamma_j^{(d)}\} & \rightarrow & \{\mathbf{D}_j^{(d)}\} & \rightarrow & \{\mathbf{U}_j^{(d)}\} \end{array} .$$

Exercise 2. Find all of the truncations $\hat{f}_j^{(d)}$ and their cones $\mathbf{U}_j^{(d)}$ for the following polynomials f:

1) $x_1 + x_2 + x_1 x_2$;
2) $x_1^3 + x_2^3 - 3x_1 x_2 + x_1 x_2^2$;
3) $x_1^2 + x_2^2 + x_1 x_2$.

This entire construction is applicable to infinite sums $f(x_1, x_2)$. For example, if f is a series (1) with integral, non-negative exponents, evaluated along curves (9) which approach the origin, then $P < 0$ and, in agreement with Section 1.5, the number of truncations $\hat{f}$ is finite. They correspond to the vertices and edges of the part of $\partial\Gamma$ between the vertices Q^* and Q_*. Formula (13) also applies. Note that the truncation $\hat{f}_j^{(0)}$, corresponding to the vertex $\Gamma_j^{(0)} = Q$, is the monomial $f_{q_1 q_2} x_1^{q_1} x_2^{q_2}$. But the truncation $\hat{f}_j^{(1)}$, corresponding to an edge of $\partial\Gamma$, will be a quasi-homogeneous polynomial, since all of the exponent vectors Q lie on a single line.

2.6. Power Transformations

Let

$$\alpha = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}$$

be a matrix with real elements and $\det \alpha \neq 0$. Then the *power transformation*

$$\begin{aligned} y_1 &= x_1^{\alpha_{11}} x_2^{\alpha_{12}} , \\ y_2 &= x_1^{\alpha_{21}} x_2^{\alpha_{22}} \end{aligned} \tag{15}$$

has the inverse

$$\begin{aligned} x_1 &= y_1^{\beta_{11}} y_2^{\beta_{12}} , \\ x_2 &= y_1^{\beta_{21}} y_2^{\beta_{22}} , \end{aligned} \qquad \beta = \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix} , \tag{16}$$

where $\beta = \alpha^{-1}$. In fact, transformations (15) and (16) are linear with respect to the logarithms of the coordinates:

$$\begin{cases} \ln y_1 = \alpha_{11} \ln x_1 + \alpha_{12} \ln x_2 , \\ \ln y_2 = \alpha_{21} \ln x_1 + \alpha_{22} \ln x_2 ; \end{cases}$$

$$\begin{cases} \ln x_1 = \beta_{11} \ln y_1 + \beta_{12} \ln y_2 , \\ \ln x_2 = \beta_{21} \ln y_1 + \beta_{22} \ln y_2 . \end{cases}$$

and are clearly mutually inverse. We now discuss some properties of the power transformation (15).

1) Transformation (15) is always well-defined and single-valued in the first quadrant of the $\mathbf{R}_0^2$ plane with coordinates x_1, x_2 or y_1, y_2. In the other quadrants, the transformation is real only under certain restrictions on the α_{ij}. The transformation is always well defined on the complex plane $\mathbf{C}_0^2$, but it is not always single-valued there.

2) Under transformation (15), the monomial $x_1^{q_1}x_2^{q_2}$ transforms to

$$x_1^{q_1}x_2^{q_2} = (y_1^{\beta_{11}}y_2^{\beta_{12}})^{q_1}(y_1^{\beta_{21}}y_2^{\beta_{22}})^{q_2} = y_1^{\beta_{11}q_1+\beta_{21}q_2}y_2^{\beta_{12}q_1+\beta_{22}q_2} \ . \tag{17}$$

where β is given in (16). That is, the vector exponent Q of X transforms into the vector exponent

$$Q' = \beta^* Q \tag{18}$$

of $Y = (y_1, y_2)$, where β^* is the transpose of matrix β. Thus, under the non-linear transformation (15), the vector exponents undergo a linear transformation. This holds for all of the terms in a sum of monomials

$$f = \sum_{Q \in \mathbf{D}} f_Q x_1^{q_1}x_2^{q_2} \ , \tag{19}$$

which, under transformation (15), becomes the sum

$$f' = \sum_{Q' \in \mathbf{D}'} f_Q y_1^{q_1'}y_2^{q_2'} \ , \tag{20}$$

where $Q' = \beta^* Q$ and $\mathbf{D}' = \beta^* \mathbf{D}$. That is, the power transformation (15) leads to a linear transformation (18) in the $\mathbf{R}_1^2$ plane of vector exponents Q. As we saw in § 1, the geometrical constructions of sets $\mathbf{D}$ and $\mathbf{D}'$ are related by that linear transformation; the polygon $\Gamma(\mathbf{D}')$ is just $\Gamma' = \beta^*\Gamma$, and each of the vertices or edges $\Gamma_j^{(d)} \subset \Gamma$ corresponds to a vertex or edge $\Gamma_j^{(d)'} \subset \Gamma'$. Therefore, the truncation $\hat{f}_j^{(d)}(X)$ transforms under (15) into a truncation $\hat{f}_j^{(d)'}$; that is, the operation of truncating sums commutes with the power transformation.

3) Under transformation (15), the curve of the class $\mathscr{W}$

$$\mathscr{F}: \begin{cases} x_1 = b_1\tau^{p_1}(1 + o(1)) \ , \\ x_2 = b_2\tau^{p_2}(1 + o(1)) \end{cases}$$

with positive b_1 and b_2, transforms into the curve

$$y_1 = (b_1\tau^{p_1})^{\alpha_{11}}(b_2\tau^{p_2})^{\alpha_{12}}(1 + o(1)) = b_1^{\alpha_{11}}b_2^{\alpha_{12}}\tau^{\alpha_{11}p_1+\alpha_{12}p_2}(1 + o(1)) \ ,$$

$$y_2 = (b_1\tau^{p_1})^{\alpha_{21}}(b_2\tau^{p_2})^{\alpha_{22}}(1 + o(1)) = b_1^{\alpha_{21}}b_2^{\alpha_{22}}\tau^{\alpha_{21}p_1+\alpha_{22}p_2}(1 + o(1))$$

This curve is also of the class $\mathscr{W}$, but with the vector order

$$P' = \alpha P \ . \tag{21}$$

Thus, the power transformation (15) induces a linear transformation (21) in the $\mathbf{R}_2^2$ plane of vector orders P. This transformation is the dual of the transformation (18). The $\mathbf{R}_1^2$ and $\mathbf{R}_2^2$ planes are thus dual spaces, and the scalar product $\langle Q, P\rangle$ is preserved under the power transformation. In particular, the cones of truncations $\mathbf{U}_j^{(d)}$ of sum (19) are transformed into the cones of truncations $\mathbf{U}_j^{(d)'}$ of sum (20). In general, all geometrical objects in the dual planes $\mathbf{R}_1^2$ and $\mathbf{R}_2^2$ undergo linear transformations (18) and (21), and all of their mutual properties are preserved.

In what follows, we will mainly use power transformations of two kinds: those for which the matrix α is triangular or unimodular. We therefore present here a few properties of such transformations.

The First Kind: Let the matrix α be triangular:

$$\alpha = \begin{pmatrix} 1 & 0 \\ \alpha_{21} & 1 \end{pmatrix},$$

where α_{21} is rational, i.e., $\alpha_{21} = n/m$. Then

$$\beta = \alpha^{-1} = \begin{pmatrix} 1 & 0 \\ -\alpha_{21} & 1 \end{pmatrix}.$$

If m, the denominator of α_{21}, is odd, then transformation (15) is defined and one-to-one for all real, non-zero x_1, x_2, y_1, y_2 (i.e., in the $\mathbf{R}_0^2$ plane). If m is even, then the real transformation (15) is only defined for $x_1 > 0$ (or $y_1 > 0$). In this half plane, it is one-to-one everywhere but on the coordinate axis $x_2 = 0$ (or $y_2 = 0$).

In the complex plane $\mathbf{C}_0^2$, the transformation (15) is one-to-one only for integral values of α_{21}.

The Second Kind: The matrix α is unimodular (i.e., the α_{ij} are integers and $\det \alpha = \pm 1$, see section 1.9). Such power transformations (15) are one-to-one in real and in complex domains away from the coordinate axes; this is the distinguishing property of power transformations with unimodular matrices α.

Moreover, any monomial $x_1^{q_1} x_2^{q_2}$ with integral exponents q_1 and q_2 transforms into a monomial $y_1^{q_1'} y_2^{q_2'}$, again with integral exponents q_1' and q_2'. This also holds for the inverse transformation. This is because unimodular transformations (18) map the integral lattice $\mathbf{Z}^2$ onto itself.

Power transformations were introduced and studied by the author [Bruno, 1962, 1965] for multi-dimensional problems. In the same papers, polygons (polyhedra) and cones of truncation for systems of differential equations were introduced. In the first version of the second paper [Bruno, 1965], these topics were treated in depth, but editorial requirements forced the article to be cut by one third, and the examples and geometrical interpretations were removed. Some of these were later restored, and can be found in the author's preprint [Bruno, 1973b].

2.7. The Critical Point

We return now to finding solutions of equation (2) in the neighborhood of the critical point $x_1 = x_2 = 0$. Any number of branches of the curve made up of solutions of equation (2) may pass through the critical point. We need to know how to distinguish these branches and how to find them with an arbitrary degree of accuracy.

Remark. We can assume that the coordinate axes $x_1 = 0$ and $x_2 = 0$ are not solutions of equation (2). In fact, if this is not so, then $f = f_0 x_1^{r_1} x_2^{r_2}$, where r_1 and r_2 are non-negative integers, and f_0 is a function, analytic at the origin, which has no factors of the form x_1 or x_2. Then the solutions to equation (2) satisfy one of the equations

$$x_1^{r_1} = 0 \ , \qquad x_2^{r_2} = 0 \ , \qquad f_0 = 0 \ .$$

The first two have trivial solutions, and we proceed to solve the third equation, which does not have the axes as solutions.

We will seek solutions of equation (2) in the form of curves (9) of the class $\mathscr{W}$. Clearly, we are only interested in those curves which approach the origin as $\tau \to \infty$; that is, those for which $P < 0$.

The third quadrant $\{P < 0\}$ of the $\mathbf{R}_2^2$ plane is divided up into the normal cones $\mathbf{U}_j^{(d)}$ of the vertices and edges of the Newton open polygon $\hat{\Gamma}$ for sum (1). Let $P \in \mathbf{U}_j^{(d)}$; then, in accordance with (13), on curve (9) we have

$$f_{\mathscr{F}} = \hat{f}_j^{(d)}(b_1, b_2)\tau^r + \tau^r o(1) \ .$$

Consequently, as $\tau \to \infty$ along the curve (9), the ratio $f_{\mathscr{F}}/\tau^r$ must approach the value $\hat{f}_j^{(d)}(b_1, b_2)$. If the curve (9) satisfies equation (2), then this value must vanish. That is, the coefficients b_1 and b_2 must satisfy the equation

$$\hat{f}_j^{(d)}(b_1, b_2) = 0 \ , \qquad b_1 \neq 0 \ , \qquad b_2 \neq 0 \ . \tag{22}$$

If $d = 0$ (that is, $\Gamma_j^{(d)}$ is a vertex), then the truncation $\hat{f}_j^{(d)}$ consists of a single term: $\hat{f}_j^{(0)} = a x_1^{q_1} x_2^{q_2}$. In this case, equation (22) has the form $a b_1^{q_1} b_2^{q_2} = 0$, and has no non-zero solutions. Thus, no curve (9) whose vector order, P, lies in the cone of vertex $\mathbf{U}_j^{(0)}$ can be a solution of equation (2). That is, vertices do not provide us with solutions, and we must turn in our search for solutions to the edges $\Gamma_j^{(1)}$ of Γ.

If $d = 1$, then $\hat{f}_j^{(d)}(x_1, x_2)$ is a quasi-homogeneous polynomial. We can exploit this quasi-homogeneity of $\hat{f}$ by applying a power transformation (15) under which the edge $\Gamma_j^{(1)}$ becomes an edge $\Gamma_j^{(1)\prime}$, parallel to a coordinate axis. We considered such a problem at the end of § 1. The vector $P' = \alpha P$ will lie on a coordinate axis. The power transformation matrix α may be taken to be either triangular (Section 1.8) or unimodular (Section 1.9). Corresponding to the kind of the power transformations used, there are two different methods of solving equation (2).

The *first method* employs power transformations with triangular matrices α, and seeks a solution of the form

$$\begin{cases} x_1 = \sigma\tau^{-1} \ , \\ x_2 = b^{(1)}\tau^{-p^{(1)}} + b^{(2)}\tau^{-p^{(2)}} + \cdots \ , \end{cases} \tag{23}$$

where the numbers $p^{(k)}$ are positive, rational, and increasing with increasing k,

and $\sigma = \pm 1$. If all of the denominators of the $p^{(k)}$ are odd, then in solution (23), σ may take on either of its possible values; that is, the solution is defined for both positive and negative x_1. If any of the $p^{(k)}$ has an even denominator, however, the sign of σ and x_1 is fixed.

We must show that every branch of the solutions of equation (2) can be represented in this form, and that distinct branches have distinct expansions. The differences may only be in terms of very high order in the expansion (23). Then those branches for which the initial terms of (23) coincide lie very near each other in a neighborhood of the origin.

The *second method* employs power transformations with unimodular matrices α and leads to expansions of the form

$$x_i = b_i^{(1)}\tau^{p_i^{(1)}} + b_i^{(2)}\tau^{p_i^{(2)}} + \cdots , \qquad i = 1, 2 , \tag{24}$$

where the $p_i^{(k)}$ are negative integers.

2.8. The First Method

We seek a curve

$$\mathscr{F}: \begin{cases} x_1 = \sigma\tau^{-1} , \\ x_2 = b^{(1)}\tau^{-p^{(1)}}(1 + o(1)) \end{cases} \tag{25}$$

of the class $\mathscr{W}$ which satisfies equation (2). Here $P = (-1, -p^{(1)})$, $b_1 = \sigma = \pm 1$, $b_2 = b^{(1)} \neq 0$. Equation (22) takes the form

$$\hat{f}(\sigma, b^{(1)}) = 0 . \tag{26}$$

For each edge $\Gamma_j^{(1)}$, we define the unit vector $R_j = (r_{1j}, r_{2j})$ to be the difference between neighboring integral points on that edge. We shall assume for definiteness that $r_{1j} \leqslant 0$ and $r_{2j} \geqslant 0$. Then the numbers r_{1j} and r_{2j} are relatively prime, and $kR_j = \Gamma_{j+1}^{(0)} - \Gamma_j^{(0)}$, where the positive integer k is the greatest common divisor of the coordinates of the vector $\Gamma_{j+1}^{(0)} - \Gamma_j^{(0)}$. Further, let the vector $P_j = (-r_{2j}, r_{1j})$. P_j is orthogonal to the edge $\Gamma_j^{(1)}$ and lies in the normal cone $\mathbf{U}_j^{(1)}$. Finally, we introduce the number $\gamma_j = -r_{1j}/r_{2j}$; then the vector $P = (-1, -\gamma_j) \in \mathbf{U}_j^{(1)}$ and $\gamma_{j-1} > \gamma_j > 0$.

Thus, the edge $\Gamma_j^{(1)}$ corresponds to a unique exponent $p^{(1)} = \gamma_j$ in expansion (25). Since the coordinates of R_j are integers, $p^{(1)}$ must be rational. We fix σ (for definiteness let $\sigma = +1$) and find all the nonzero roots $b_1^{(1)}, \ldots, b_s^{(1)}$ of equation (26).

In this connection, if we are interested only in the real roots of equation (2), we can limit ourselves to finding the real roots of equation (26). Let $b_*^{(1)}$ be such a root. Then the first term of expansion (23) is already known. In order to find the second term, we make the substitution

$$x_2 = x_1^{p^{(1)}}(b_*^{(1)} + y) . \tag{27}$$

Then $f(x_1, x_2) = g(x_1, y)$, where g is a series of non-negative powers of x_1 and y, with the powers of x_1 rational and those of y integral.

Reducing the series by a factor of x_1^r, we obtain $g_0 = g/x_1^r$; we must now find those branches of the curve $g_0(x_1, y) = 0$ which pass through the origin. This is similar to our original problem, except that x_1 now appears in rational powers. If m is the denominator of $p^{(1)}$, then $x_1^{1/m}$ appears in integral powers in the series. If this quantity is considered as a new variable, then the problem is identical to that with which we began.

Now, if $\partial g_0/\partial y \neq 0$ at the origin, then the origin is a simple point and, applying theorem 1, it is possible to find a solution $y(x_1)$ of the equation $g_0(x_1, y) = 0$ which is a series in powers of $x_1^{1/m}$. If $\partial g_0/\partial y = 0$ at the origin, it is necessary to construct a Newton open polygon for g_0, select edges, find the exponent p and the coefficients b for each edge, and so forth.

If the rational number $p^{(1)}$ has an odd denominator, the quantity $\tau^{p^{(1)}}$ is defined for negative as well as positive τ. It is therefore sufficient to consider the case of $\sigma = 1$, since the signs of x_1 and τ take on both values (± 1). Solutions $b^{(1)}$ of equation (26) do not depend on σ. If, however, the denominator of $p^{(1)}$ is even, then $\tau^{p^{(1)}}$ is a real quantity only for non-negative τ. It is then necessary to consider separately the cases $x_1 > 0$ and $x_1 < 0$, corresponding to $\sigma = 1$ and $\sigma = -1$. In the second case ($\sigma = -1$), it is convenient to change the sign of x_1 by writing $\tilde{x}_1 = -x_1$. In this case the solutions $b^{(1)}$ to equation (26) will be different for different σ.

In order to relate the Newton polygons $\Gamma(f)$ and $\Gamma(g)$, we split transformation (27) into two parts:

$$x_2 = x_1^{p^{(1)}} z \ , \tag{28}$$

$$z = b^{(1)} + y \ . \tag{29}$$

Under transformation (28), every monomial $ax_1^{q_1}x_2^{q_2}$ becomes a term $ax_1^{q_1+q_2p^{(1)}}z^{q_2} = ax_1^{q_1'}z^{q_2'}$. That is, the series

$$f = \sum f_{q_1q_2} x_1^{q_1} x_2^{q_2}$$

is transformed into a series

$$f = \sum f_{q_1'q_2'} x_1^{q_1'} z^{q_2'} \ ,$$

where $q_1' = q_1 + q_2 p^{(1)}$ and $q_2' = q_2$. Thus, the set $\mathbf{D}(f')$ is the image of the set $\mathbf{D}(f)$ under the linear transformation $Q' = \alpha^{*-1}Q$, where the matrix

$$\alpha^{*-1} = \begin{pmatrix} 1 & p^{(1)} \\ 0 & 1 \end{pmatrix} .$$

Under this transformation, all the points Q are shifted parallel to the q_1 axis, to the right for $q_2 > 0$. The polygon $\Gamma(f)$ is deformed into a polygon $\Gamma(f')$ whose edges and vertices are, respectively, the images of the edges and vertices

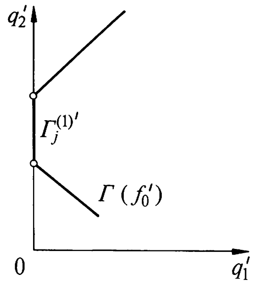

Fig. 16

of $\Gamma(f)$. In particular, points Q lying on the edge $\Gamma_j^{(1)}$, which corresponds to a truncated series $\hat{f}$, satisfy

$$\langle Q, P\rangle = \langle \Gamma_j^{(0)}, P\rangle = r \ ,$$

i.e., $q_1 + q_2 p^{(1)} = r$. After transformation (28) is applied, the points Q transform to

$$Q' = (q_1 + q_2 p^{(1)}, q_2) = (r, q_2) \ ,$$

that is, the edge $\Gamma_j^{(1)}$ becomes the vertical edge $\Gamma_j^{(1)'}$ along which $q_1' = r$, and the entire set $\mathbf{D}(f')$ lies to the right of the vertical $q_1' = r$. If we now let $f_0' = x_1^{-r} f'$, then the points Q' corresponding to the truncation $\hat{f}_0'$ of f_0' all lie on the q_2' axis. Figures 10, 11 and 16 illustrate the Newton polygons $\Gamma(f)$, $\Gamma(f')$, and $\Gamma(f_0')$, as well as the disposition of the edge $\Gamma_j^{(1)}$ which corresponds to $\hat{f}$. Here, the truncation $\hat{f}$ of the function f with respect to the vector order P is transformed into a truncation $\hat{f}_0'$ of f_0' with respect to the vector order $(-1,0)$. This $\hat{f}_0'$ depends only on z: $\hat{f}_0'(z) = x_1^{-r}\hat{f}(x_1, x_1^{p^{(1)}}z) = \hat{f}(1,z)$. Hence, the root $b^{(1)} \neq 0$ of equation (26) is just a root of the equation

$$\hat{f}_0'(z) = 0 \tag{30}$$

and determines those points on the z axis which satisfy the equation

$$f_0'(x_1, z) = 0 \ . \tag{31}$$

Each remaining point on the z axis has some neighborhood in which equation (31) has no solutions.

Note that under transformation (28), the point $x_1 = x_2 = 0$ blows up into a line, the z-axis. The curve $x_2 = bx_1^p$ which approaches the point $x_1 = x_2 = 0$ transforms into a curve $z = bx_1^{p-p^{(1)}}$, which tends to infinity, to a constant, or to zero, if $p < p^{(1)}$, $p = p^{(1)}$, or $p > p^{(1)}$, respectively, as $x_1 \to 0$.

If $b_*^{(1)}$ is a simple root of equation (26) or (30), then $d\hat{f}_0'(b_*^{(1)})/dz \neq 0$, and since

$$\partial f_0'(0, b_*^{(1)})/\partial z = d\hat{f}_0'(b_*^{(1)})/dz \ ,$$

the point $x_1 = 0$, $z = b_*^{(1)}$ is a simple point of curve (31). In agreement with theorem 1 the solutions of equation (31) in a neighborhood of the point $x_1 = 0$, $z = b_*^{(1)}$ will take the form of an expansion of $y = z - b_*^{(1)}$ in powers of $x_1^{1/m}$. This expansion gives a branch of the solutions of equation (2) in the form of the series (23).

If $b_*^{(1)}$ is not a simple root of equation (30), we must make the variable change $z = b_*^{(1)} + y$, construct the Newton polygon for the function $g_0(x_1, y) = f_0'(x_1, b_*^{(1)} + y)$, and proceed to separate the branches, as above. This will also lead to expression (23).

If equation (30) has no real root $b^{(1)}$, then the solutions of equation (2) have no real branches of the form (23) for the given order $p^{(1)}$ (i.e., they have only complex branches).

Example 3. Let

$$f(x_1, x_2) = x_1^3 + x_2^3 - 2x_1x_2 = 0 \ . \tag{32}$$

The curve defined by this equation is called the folium of Descartes. At the origin, $\partial f/\partial x_1 = \partial f/\partial x_2 = 0$; it is therefore a critical point. We will investigate our equation in a neighborhood of this point. Here $Q_1 = (3,0)$, $Q_2 = (0,3)$, and $Q_3 = (1,1)$; the Newton open polygon (fig. 17) has two edges, $\Gamma_1^{(1)}$ and $\Gamma_2^{(1)}$ (see example 7 of § 1). We begin with the edge $\Gamma_1^{(1)} \supset Q_1, Q_3$. Here $Q_3 - Q_1 = (-2, 1) = R_1$, so $P_1 = (-1, -2)$ and $\gamma = 2$. The corresponding truncated sum is $\hat{f} = x^3 - 2x_1x_2$. To find the coefficient $b^{(1)}$ we solve equation (26):

$$\hat{f}(1, b) = 1 - 2b = 0 \ ,$$

which gives us $b^{(1)} = 1/2$. This equation is linear with respect to b, and $b^{(1)}$ is a simple root. Therefore to this root and in general to the edge $\Gamma_1^{(1)}$ there correspond a simple branch of the solutions of equations (32):

$$x_2 = \tfrac{1}{2}x_1^2(1 + o(1)) \ .$$

In order to find further terms in the expansion of x_2 in powers of x_1 on this

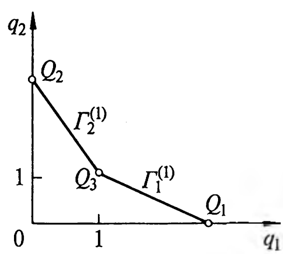

Fig. 17

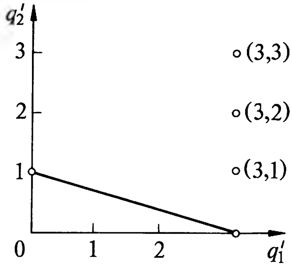

Fig. 18

branch, we must make the variable change (27)

$$x_2 = x_1^2(\tfrac{1}{2} + y) \ . \tag{33}$$

This gives

$$\begin{aligned} f(x_1, x_2) &= g(x_1, y) \\ &= x_1^3 + x_1^6(\tfrac{1}{8} + \tfrac{3}{4}y + \tfrac{3}{2}y^2 + y^3) - 2x_1^3(\tfrac{1}{2} + y) \\ &= -2x_1^3 y + \tfrac{1}{8}x_1^6 + \tfrac{3}{4}x_1^6 y + \tfrac{3}{2}x_1^6 y^2 + x_1^6 y^3 \ . \end{aligned}$$

Dividing by x_1^3, we obtain

$$g_0(x_1, y) = -2y + \tfrac{1}{8}x_1^3 + \tfrac{3}{4}x_1^3 y + \tfrac{3}{2}x_1^3 y^2 + x_1^3 y^3 \ .$$

The set $\mathbf{D}(g_0)$ and the open polygon $\partial\Gamma(g_0)$ are illustrated in figure 18. Here the point $x_1 = y = 0$ is simple, since at that point $\partial g_0/\partial y = -2 \neq 0$ (the simpleness of this point follows also from the fact that $b^{(1)} = 1/2$ is a simple root). According to theorem 1, solutions of the equation $g_0(x_1, y) = 0$ can be written as a series $y = y(x_1)$. Specifically, the first term of this series can be found from the truncation $\hat{g}_0 = -2y + (1/8)x_1^3$, corresponding to the only edge of the open polygon $\hat{\Gamma}(g_0)$. From the truncated equation $\hat{g}_0 = 0$ we find that $y = (1/16)x_1^3$; therefore, for the solution of the whole equation we have $y = (1/16)x_1^3(1 + o(1))$. Substituting this into equation (33), we obtain the expansion

$$x_2 = \tfrac{1}{2}x_1^2 + \tfrac{1}{16}x_1^5 + \cdots$$

for a branch $\mathscr{F}_1$ of the solutions of equation (32). The edge $\Gamma_1^{(1)}$ corresponds to this single branch.

Now let us consider the edge $\Gamma_2^{(1)} \supset Q_2, Q_3$. Here, $R_2 = Q_2 - Q_3 = (-1, 2)$, $P_2 = (-2, -1)$, $\gamma = 1/2$. For the coefficient $b^{(1)}$, with $\sigma = 1$, we have the equation

$$\hat{f}_2^{(1)}(1, b) = b^3 - 2b = 0 \ .$$

The roots of this equation are $b = 0$ and $b^{(1)} = \pm\sqrt{2}$. We are only interested

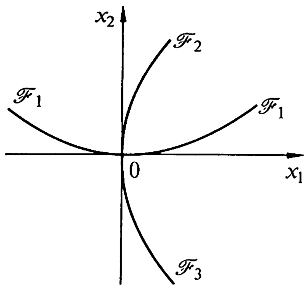

Fig. 19

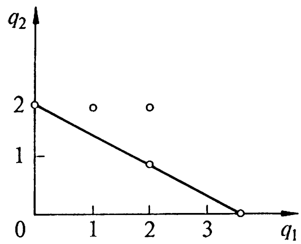

Fig. 20

in non-zero roots, so we take $b^{(1)} = \pm\sqrt{2}$. Each of these roots is simple. We therefore obtain the two branches:

$$\mathscr{F}_2\colon x_2 = \sqrt{2x_1} + \cdots\,; \qquad \mathscr{F}_3\colon x_2 = -\sqrt{2x_1} + \cdots\,.$$

For real x_1, x_2, these curves are only defined for $x_1 \geqslant 0$. For $\sigma = -1$, equation (26) becomes $b^3 + 2b = 0$, which has no real, non-zero roots. Therefore edge $\Gamma_2^{(1)}$ does not yield real branches for negative x_1. The solutions of equation (32) in the x_1, x_2 plane are shown in figure 19.

Remark. In calculations, it is sometimes easier to replace the nonlinear substitution (27) by the linear substitution $x_2 = b^{(1)}x_1^{p(1)} + y$, and then to seek such expansions of y whose first term is a power of x_1 greater than $p^{(1)}$.

Example 4. Let $f = x^4 + x^2y^2 - 2x^2y - xy^2 + y^2 = 0$ (here $x = x_1$ and $y = x_2$). The Newton open polygon (figure 20) consists of a single edge, with $R = (-2, 1)$, $P = (-1, -2)$, $\gamma = 2$, $\hat{f} = x^4 - 2x^2y + y^2$. To find the coefficient b we have the equation $\hat{f}(1, b) = 1 - 2b + b^2 = 0$, which has the double root $b = 1$. Recalling our remark above, we make the substitution $y = x^2 + z$. Then

$$f(x, y) = g(x, z) = z^2 - x^5 - 2x^3z - xz^2 + x^6 + 2x^4z + x^2z^2\ .$$

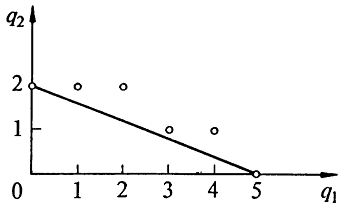

Fig. 21

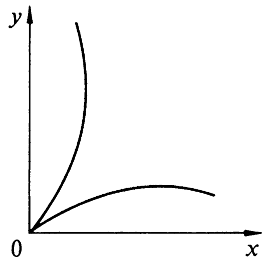

Fig. 22

The open polygon $\hat{\Gamma}(g)$ consists of a single edge (figure 21) for which $R = (-5, 2)$ and $P = (-2, -5)$, $\gamma = 5/2$, $\hat{g} = z^2 - x^5$. From the equation $\hat{g}(\sigma, b) = b^2 - \sigma = 0$, we find that $b = \pm 1$ for $\sigma = 1$; both roots are simple. There are no real roots for $\sigma = -1$. We therefore have the two simple branches

$$z = \pm x^{5/2} + \cdots, \qquad x > 0,$$

or, returning to y,

$$y = x \pm x^{5/2} + \cdots$$

The two branches are now distinguished from one another, that is, we have obtained an expansion of y in powers of x for each branch which is different from the expansion for the other branch. Note that in our first step (when we just had the double root $b = 1$) the two branches were not distinguished from one another, since the first term of each expansion is the same for both branches. The branches' behavior in the x, y plane is illustrated in figure 22.

Exercise 3. Near the origin find and sketch the real branches of the solutions of the following equations:

1) $x^3 - x^2 + y^2 = 0$,
2) $x^3 + x^2 + y^2 = 0$,
3) $x^3 + y^2 + x^2y^2 = 0$,
4) $2x^4 - 3x^2y + y^2 - 2y^3 + y^4 = 0$,

5) $(x^2 + y^2)^2 + 3x^2y - y^3 = 0$,
6) $(x^2 + y^2)^3 - 4x^2y^2 = 0$,
7) $y^6 - x^3y^2 - x^5 = 0$,
8) $x^2y - 2xy^2 + y^3 + x^4 + 2x^3y - 2xy^3 - y^4 + ax^5 = 0$ for $a = 15, 16, 17$,
9) $-x^3 + x^4 - 2x^2y - xy^2 + 2xy^4 + y^5 = 0$.

The method presented here goes back to Newton [1937] (see Chebotar'ev, 1943, Walker, 1950, Vaĭnberg and Trenogin, 1969, where different properties of the expansions resulting from this method are described). The method has one disadvantage: it leads to expansions in fractional powers of x_1. In order to avoid this defect, Briot and Bouquet [1856] suggested that (18) be replaced by the following change of variables:

$$x_1 = u^m , \qquad x_2 = u^n(b^{(1)} + v) , \tag{34}$$

where the integers m and n are, respectively, the denominator and numerator of the fraction $p^{(1)}$. After such a transformation, a series $f(x_1, x_2)$ becomes a series in integral powers of u and v. However, the inverse of transformation (34) generally includes fractional powers. That is, transformation (34) is not one-to-one away from the coordinate axes in the complex plane $\mathbf{C}_0^2$. Therefore, we now turn our attention to a second method of solving equation (2), a method based on power transformations with unimodular matrices.

2.9. The Second Method of Resolution of a Singularity

As above, let $R_j = (r_{1j}, r_{2j})$ denote the unit vector of edge $\Gamma_j^{(1)}$. Then the vector $P_j = (-r_{2j}, r_{1j})$ lies in the normal cone $\mathbf{U}_j^{(1)}$ and is an integral vector. Let α be a unimodular matrix, such that

$$P' = \alpha P_j = (-1, 0) .$$

In Section 1.9 we showed that

$$\alpha = \begin{pmatrix} s_1 & s_2 \\ r_1 & r_2 \end{pmatrix} , \tag{35}$$

where the integers s_1 and s_2 are chosen to satisfy

$$s_1 r_2 - s_2 r_1 = 1 . \tag{36}$$

For any pair of relatively prime integers r_1 and r_2, there is an infinite number of pairs of integers s_1 and s_2 which satisfy equation (36). The inverse matrix is

$$\beta = \alpha^{-1} = \begin{pmatrix} r_2 & -s_2 \\ -r_1 & s_1 \end{pmatrix} . \tag{37}$$

We carry out coordinate change (15) with matrix α and, in agreement with the formulae of Section 2.6, we have

$$x_1^{q_1} x_2^{q_2} = y_1^{r_2 q_1 - r_1 q_2} y_2^{-s_2 q_1 + s_1 q_2} = y_1^{q_1'} y_2^{q_2'} ,$$

that is, the exponent of y_1' is

$$q_1' = -\langle Q, P_j \rangle \geqslant r = -\max_{Q \in \mathbf{D}} \langle Q, P_j \rangle .$$

Sum (19) is transformed into sum (20), and the polygon $\Gamma = \Gamma(f)$ becomes the polygon $\Gamma' = \Gamma(f')$. In particular, the edge $\Gamma_j^{(1)}$ becomes an edge $\Gamma_j^{(1)'}$ orthogonal to the vector $P' = (-1, 0)$. Consequently, $\Gamma_j^{(1)'}$ is parallel to the ordinate axis, and the first coordinate of every point $Q \in \Gamma_j^{(1)'}$ is the same number r (figs. 10, 11). For the corresponding truncated sums we obtain

$$\hat{f}_j^{(1)}(x_1, x_2) = \hat{f}_j^{(1)'}(y_1, y_2) = y_1^r \hat{f}_0'(y_2) .$$

For the remaining terms of expansion (20) $q_1' > r$. Therefore, the function $f'(y_1, y_2)$ may be divided by y_1^r:

$$f_0'(y_1, y_2) = y_1^{-r} f'(y_1, y_2) .$$

The division of $f'(y_1, y_2)$ by y_1^r corresponds to shifting the polygon Γ' to the left a distance r in the $\mathbf{R}_1^2$ plane, so that the edge $\Gamma_j^{(1)'}$ lies along the q_2' axis (figure 16).

We have now arrived at the following problem: find the solutions of the equation

$$f_0'(y_1, y_2) = 0 \tag{38}$$

as class $\mathscr{W}$ curves of vector order $P = (-1, 0)$

$$y_1 = b_1' \tau^{-1}(1 + o(1)) ,$$

$$y_2 = b_2'(1 + o(1)) , \qquad b_2' \neq 0, \infty .$$

The truncated series for this vector order is $\hat{f}_0' = \hat{f}_0'(y_2)$. That is, we obtain the following equation for the coefficient b_2':

$$\hat{f}_0'(b_2') = 0 . \tag{39}$$

Here, we are interested in the finite, non-zero roots of this equation. If $b_2' = b_{2*}'$ is such a root, we must find solutions of equation (38) near the point $y_1 = 0$, $y_2 = b_{2*}'$. As long as f_0' is expressed in terms of integral powers of y_1 and y_2, this is precisely the problem with which we started out: find all solutions of an analytic equation in some neighborhood of a known solution.

If $b_2' = b_{2*}'$ is a simple root of (39) (i.e., $d\hat{f}_0'/dy_2 \neq 0$), then, by theorem 1, all solutions of equation (38) near the point $y_1 = 0$, $y_2 = b_{2*}'$ are given by a series $b(y_1)$ in positive integral powers of y_1: $y_2 = b_{2*}' + b(y_1)$. This branch of the solutions of equation (38) is separated from other branches. Returning to x_1, x_2 coordinates with the help of the inverse power transformation (16), we obtain expressions for x_1 and x_2 in the form of series in powers of y_1. That is, the solution

of equation (2) will be represented in parametric form (24) with the parameter τ equal to y_1^{-1}.

If $b_2' = b_{2*}'$ is a multiple root of equation (39), then the point $y_1 = 0$, $y_2 = b_{2*}'$ may be a critical point of the equation $f_0'(y_1, y_2)$. To examine this point, we make the translation $z = y_2 - b_{2*}'$, construct the Newton open polygon for the function

$$g(y_1, z) = f_0'(y_1, y_2)$$

select edges, etc. In other words, we must investigate the function $g(y_1, z)$ in exactly the same manner described here for the function $f(x_1, x_2)$.

It remains to be shown that a finite number of such steps is sufficient to distinguish all branches of solutions of equation (2) from each other. For an edge $\Gamma_j^{(1)}$, we call one less than the number of integral points on the edge the *height*, which we denote by δ_j. The sum of the heights of all edges of $\hat{\Gamma}$ is the *height of the open polygon* $\hat{\Gamma}$, denoted by $\delta = \delta(\hat{\Gamma})$. The number δ characterizes the degree of the singularity at the origin. If $f = 0$ at the origin and $\partial f / \partial x_2 \neq 0$, then $\delta = 1$ and the origin is simple. In general, using the extreme vertices Q^* and Q_* of the Newton open polygon, it is easy to see that

$$\delta \leqslant q_2^* - q_{2*} \,, \qquad \delta \leqslant q_{1*} - q_1^* \,. \tag{40}$$

We now note that the height δ_j of an edge is equal to the height δ of the entire open polygon only if the open polygon $\hat{\Gamma}$ consists solely of that edge; otherwise, $\delta_j < \delta$. Further, under the power transformation (15) with a unimodular matrix α, there is a one-to-one correspondence between the points of the integral lattices of the q_1, q_2 and q_1', q_2' planes. Thus, the height of an edge is invariant under those transformations.

If we divide $\hat{f}_0'(y_2)$ by the factor y_2^s of maximal degree (i.e., $f_0'(y_2) = y_2^s h(y_2)$ with $h(0) \neq 0$), then the height of $\Gamma_j^{(1)\prime}$ is equal to the degree of $h(y_2)$ (figures 16 and 17).

In order to investigate the neighborhood of the point $y_1 = 0$, $y_2 = y_2^0$, we construct the Newton open polygon $\hat{\Gamma}(g(y_1, z))$. According to (40), the polygon's height does not exceed the coordinate q_2^*. In turn, the number q_2^* is the multiplicity of the root $b_2' = y_2^0$ of equation (39). But the multiplicity of this root $y_2^0 \neq 0$ does not exceed the degree of the polynomial $h(y_2)$, which equals the height δ_j of the edge $\Gamma_j^{(1)}$. Thus, the height of $\hat{\Gamma}(g(y_1, z))$ cannot exceed the height of any edge of the original open polygon, $\hat{\Gamma}(f(x_1, x_2))$. Equality is possible only when the open polygon $\hat{\Gamma}(f)$ consists of a single edge and when the polynomial $\hat{f}_0'(y_2)$ has only one root $y_2^0 \neq 0$, the multiplicity of which is the degree of the polynomial $h(y_2)$. That is, in all other cases the procedure described above will result in a reduction in the height (and degree) of the singularity being investigated, and will reduce to the case of a singular point of unit height. This will be a simple point, corresponding to an isolated branch. Unit height cannot be reached in a finite number of steps, like those described above, only if equation (2) has multiple branches. For example, if

$$f = \left(x_2 - \sum_{k=1}^{\infty} x_1^k\right)^3 g(x_1, x_2) ,$$

then the branch $x_2 = \sum_{k=1}^{\infty} x_1^k$ will be a triple solution of equation (2), and by our method of calculation the height will always be 3.

We note further that the number of branches of the solutions of equation (2) does not exceed the height δ of the Newton open polygon $\hat{\Gamma}(f)$. The number of real branches (corresponding to real roots $y_2 = y_2^0$ of equations like (39)) may be even less.

Example 5. (see example 3). $f = x_1^3 + x_2^3 - 2x_1x_2$; the open polygon $\hat{\Gamma}$ is pictured in figure 17. For edge $\Gamma_1^{(1)}$, the vector R_1 is $(-2, 1)$. We seek integers s_1 and s_2 that satisfy equation (36):

$$\begin{vmatrix} s_1 & s_2 \\ -2 & 1 \end{vmatrix} = s_1 + 2s_2 = 1 .$$

Here we can use $s_1 = 1, s_2 = 0$. Then variable change (15) is:

$$y_1 = x_1 , \qquad y_2 = x_1^{-2} x_2 . \tag{41}$$

By (37) the inverse change of variables (16) is

$$x_1 = y_1 , \qquad x_2 = y_1^2 y_2 . \tag{42}$$

Transformation (41) transforms our function f into

$$f'(y_1, y_2) = f(x_1, x_2) = y_1^3 + y_1^6 y_2^3 - 2y_1^3 y_2 .$$

Dividing by y_1^3, we obtain

$$f_0' = y_1^{-3} f' = 1 - 2y_2 + y_1^3 y_2^3 ;$$

which we truncate to

$$\hat{f}_0'(y_2) = f_0'(0, y_2) = 1 - 2y_2 .$$

This polynomial has a single, simple root $y_2^0 = 1/2$; the corresponding branch is therefore isolated. Making the substitution

$$y_2 = y_2^0 + z = \tfrac{1}{2} + z ,$$

we obtain

$$f_0'(y_1, \tfrac{1}{2} + z) = -2z + y_1^3(\tfrac{1}{2} + z)^3 = -2z + \tfrac{1}{8} y_1^3 + \cdots$$

A first approximation to this is

$$-2z + \tfrac{1}{8} y_1^3 ,$$

with a root $z = (1/16)y_1^3$; thus, $z = (1/16)y_1^3 + \cdots$, where the dots denote terms which contain y_1 in powers greater than 3; hence

$$y_2 = \tfrac{1}{2} + \tfrac{1}{16}y_1^3 + \cdots$$

and, according to (42),

$$x_1 = y_1 \ , \qquad x_2 = \tfrac{1}{2}y_1^2 + \tfrac{1}{16}y_1^5 + \cdots$$

This branch of the solution curve has exactly the same form as in example 3. For the edge $\Gamma_1^{(1)}$, the transformations of both methods are the same.

We now turn to edge $\Gamma_2^{(1)}$, for which $R_2 = (-1, 2)$, and we seek integers s_1 and s_2 that satisfy

$$\begin{vmatrix} s_1 & s_2 \\ -1 & 2 \end{vmatrix} = 2s_1 + s_2 = 1 \ .$$

We could use $s_1 = 0, s_2 = 1$, but, after transformation (15), there would be negative powers of y_2 in f'. To avoid this, we choose $s_1 = 1, s_2 = -1$. Transformation (15) takes the form

$$y_1 = x_1 x_2^{-1} \ , \qquad y_2 = x_1^{-1} x_2^2 \ ,$$

and its inverse (16) is, according to (37),

$$x_1 = y_1^2 y_2 \ , \qquad x_2 = y_1 y_2 \ . \tag{43}$$

Thus

$$\begin{aligned} f'(y_1, y_2) &= f(x_1, x_2) \\ &= y_1^6 y_2^3 + y_1^3 y_2^3 - 2y_1^3 y_2^2 = y_1^3 y_2^2 (y_1^3 y_2 + y_2 - 2) \ . \end{aligned}$$

The truncation is $\hat{f}_0' = y_2^2(y_2 - 2)$, from which we obtain the root $y_2^0 = 2$. This is a simple root, so that the branch is isolated. We find the linear part of the expansion of y_2 in terms of y_1 from the equation

$$g(y_1, y_2) = y_1^3 y_2 + y_2 - 2 = 0 \ .$$

The substitution $y_2 = 2 + z$ gives

$$g(y_1, 2 + z) = z + 2y_1^3 + \cdots \ ,$$

from which we obtain

$$z = -2y_1^3 + \cdots$$

Thus, returning to our original variables, we find

$$y_2 = 2 - 2y_1^3 + \cdots \ ,$$

and, according to (43),

$$x_1 = 2y_1^2 - 2y_1^5 + \cdots \ ,$$

$$x_2 = 2y_1 - 2y_1^4 + \cdots$$

Here, y_1 is the parameter τ^{-1}. The branches are shown in figure 19.

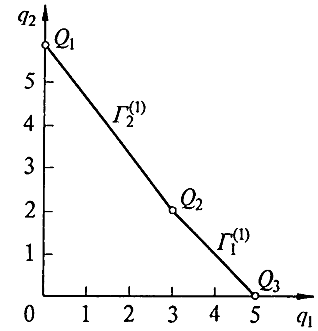

Fig. 23

Example 6. $f = x_2^6 - x_1^3 x_2^2 - x_1^5 = 0$. The open polygon $\hat{\Gamma}$ (figure 23) has two edges. For edge $\Gamma_1^{(1)}$, $Q_2 - Q_3 = (-2, 2)$, so $R_1 = (-1, 1)$, and equation (36) is satisfied by $s_1 = 1, s_2 = 0$. We thus have the power transformation

$$y_1 = x_1 \ , \qquad y_2 = x_1^{-1} x_2$$

and its inverse,

$$x_1 = y_1 \ , \qquad x_2 = y_1 y_2 \ .$$

Thus

$$f'(y_1, y_2) = y_1^6 y_2^6 - y_1^5 y_2^2 - y_1^5 = y_1^5(y_1 y_2^6 - y_2^2 - 1) \ ;$$

the roots of the truncated equation $y_1^{-5}\hat{f}' = -y_2^2 - 1 = 0$ are imaginary, $y_2^0 = \pm i$. They correspond to complex branches of the solution curve of the original equation. Since this truncated equation has no real roots, the edge $\Gamma_1^{(1)}$ corresponds only to complex branches.

For edge $\Gamma_2^{(1)}$, we have $Q_1 - Q_2 = (-3, 4) = R_2$. Expanding γ into a continued fraction, we obtain $\gamma = |r_1/r_2| = 3/4 = (1 + 1/3)^{-1}$. Equation (36) is satisfied by $s_1 = 1, s_2 = -1$. This gives us the transformation

$$y_1 = x_1 x_2^{-1} \ , \qquad y_2 = x_1^{-3} x_2^4 \ ,$$

and its inverse

$$x_1 = y_1^4 y_2 \ , \qquad x_2 = y_1^3 y_2 \ .$$

We thus obtain

$$f' = y_1^{18} y_2^6 - y_1^{18} y_2^5 - y_1^{20} y_2^5 = y_1^{18} y_2^5 (y_2 - 1 - y_1^2) \ .$$

The truncation

$$y_1^{-18}\hat{f}' = y_2^5(y_2 - 1)$$

has only one non-zero root, $y_2^0 = 1$; this is a simple root, corresponding to a simple, isolated branch of the solutions of the original equation $f = 0$. After making the substitution $y_2 = 1 + z$, we find that $z = y_1$. Unlike our previous cases, this is an exact and not an approximate solution of $f' = 0$. Thus,

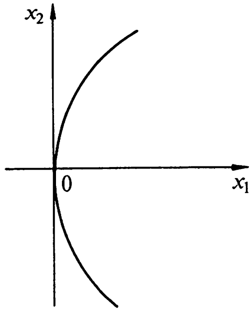

Fig. 24

$y_2 = 1 + y_1^2$, and

$$x_1 = y_1^4 + y_1^6 , \qquad x_2 = y_1^3 + y_1^5 .$$

The disposition of this solution in the x_1, x_2 plane is illustrated in figure 24.

Exercise 4. Do exercise 3 using the method of this section.

2.10. The Method of Weierstrass

Weierstrass [1902] outlined a method of solving equation (2) with the help of birational changes of coordinates. We call a transformation from x_1, x_2 to y_1, y_2 birational if the y_i can be expressed as a ratio of polynomials in x_1 and x_2, and the x_i can likewise be written as a ratio of polynomials in y_1 and y_2. Birational transformations include parallel translations, affine transformations, power transformations with unimodular matrices, as well as all (finite) superpositions of any of these. In particular, the transformations used in the preceding section to isolate branches of solutions are birational. We will describe Weierstrass' method from our point of view, which differs from [Weierstrass, 1902], as follows.

Weierstrass showed that branches of an analytic curve can be found using a finite number of transformations of the form

$$y_1 = x_1 x_2 , \qquad y_2 = x_2 \tag{44}$$

or

$$y_1 = x_1 , \qquad y_2 = x_1 x_2 \tag{45}$$

and parallel translations. In the preceding section we saw that a finite number of power transformations (15) with unimodular matrices α and parallel translations is sufficient to isolate all branches. We will show here that a power transformation with a unimodular matrix α can be expressed as a superposition of a finite number of transformations of the form of (44) or (45). These are, in fact, power

transformations with corresponding matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \tag{46}$$

Thus, we need to show that the matrices α described earlier can be expressed as finite products of matrices of form (46). A method of finding the matrix α was presented in section 1.9, where we also showed that α is a product of triangular matrices

$$\alpha_k = \begin{pmatrix} 1 & a_k \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \alpha_k = \begin{pmatrix} 1 & 0 \\ a_k & 1 \end{pmatrix} \tag{47}$$

with integers a_k. It is easy to see that

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^a = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^a = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}.$$

That is, every matrix of the form (47) is a product of matrices of the form (46). Thus, each power transformation of the previous section can be decomposed into a finite number of Weierstrass' transformations (44) and (45). A succession of transformations of form (44) and (45) is used in algebraic geometry in the resolution of singularities of algebraic curves; this is called a "*sigma-process*" (see Walker, 1950). The resulting power transformation could be called a "multiple sigma-process", since its use would require fewer calculations than the usual sigma process, with identical results.

Bendixson [1898] applied the method of Weierstrass to the resolution of the singularities of an ordinary differential equation (see also Dulac, 1934). His results could also be obtained with the transformations of our "second method", as we shall see in Chapter II.

§ 3. Level Curves of an Analytic Function

3.1. Statement of the Problem

Let the function $f(x_1, x_2)$ be defined in some region. The *level curves* of this function are the curves determined by the equation

$$f(x_1, x_2) = c = \text{const.} \tag{1}$$

Different values of the constant c correspond, of course, to different level curves. In § 3 we will consider the following **problem**: let the function $f(x_1, x_2)$ be analytic at the point $x_1 = x_1^0$, $x_2 = x_2^0$; in an arbitrarily small neighborhood $\mathscr{U}$ of this point, find explicit expressions for the level curves (1) of f:

$$x_i = b_i(c, \tau) \ , \qquad i = 1, 2 \ , \tag{2}$$

(τ is a parameter), and determine their relative positions. For the sake of visualization, we will consider here the real case when the function f and the coordinates x_i are real; however, all of the methods presented can be applied equally well to complex situations. Without any loss of generality, we can also assume that $x_1^0 = x_2^0 = 0$ and $f(x_1^0, x_2^0) = 0$. This can be accomplished by making parallel translations in the coordinates x_i and the constant c. Then the function f can be expanded in a Taylor series

$$f = \sum_{Q \geqslant 0} f_Q X^Q \ , \qquad f_0 = 0 \ , \tag{3}$$

which converges absolutely in some neighborhood of the point $X = (x_1, x_2) = 0$. We denote here and afterwards

$$X = (x_1, x_2) \ , \qquad Q = (q_1, q_2) \ , \qquad X^Q = x_1^{q_1} x_2^{q_2} \ .$$

3.2. Coordinate Transformations

We review here a few facts of analysis. Let there be a transformation

$$x_i = \xi_i(Y) \ , \qquad \xi_i(0) = 0 \ , \qquad i = 1, 2 \ , \tag{4}$$

which transforms the point $Y = (y_1, y_2) = 0$ into the point $X = 0$. In some neighborhood of $Y = 0$, let the functions ξ_i have continuous partial derivatives, and let the Jacobian

$$\frac{D(x_1, x_2)}{D(y_1, y_2)} = \begin{vmatrix} \dfrac{\partial \xi_1}{\partial y_1} & \dfrac{\partial \xi_1}{\partial y_2} \\ \dfrac{\partial \xi_2}{\partial y_1} & \dfrac{\partial \xi_2}{\partial y_2} \end{vmatrix}$$

be non-zero at $Y = 0$. Then transformation (4) has the following properties: (see Courant, 1931, pt. II, Ch. 3, § 3)

1) The image of every sufficiently small neighborhood of the point $Y = 0$ contains some neighborhood of the point $X = 0$. That is, for every ε in some interval $(0, \varepsilon^0)$, there exists a $\delta > 0$, such that the image of the neighborhood

$$|y_i| < \varepsilon \ , \qquad i = 1, 2 \tag{5}$$

contains the neighborhood

$$|x_i| < \delta \ , \qquad i = 1, 2 \ . \tag{6}$$

2) Transformation (4) is invertible in some neighborhood of the point $Y = 0$. That is, there exists an inverse transformation to (4),

$$y_i = \eta_i(X) \ , \qquad \eta_i(0) = 0 \ , \qquad i = 1, 2 \ , \tag{7}$$

which is defined on some neighborhood of the point $X = 0$. Transformation (7) also possesses property 1) (with X and Y exchanged).

3) There exist a neighborhood $\mathcal{U}$ of $X = 0$ and a neighborhood $\mathcal{V}$ of $Y = 0$, which lie in the domains of definition and ranges of values of transformations (4) and (7), and between, which these transformations establish a one-to-one correspondence.

Consequently, transformation (4) constitutes a one-to-one *change of coordinates* from X to Y in the region $\mathcal{U}$.

If the functions ξ_i are analytic at $Y = 0$, then they can be expanded in power series in Y, convergent in some set of form (5). Then the inverse transformation (7) will also be analytic at $X = 0$, i.e., the functions η_i can be expanded in Taylor series, convergent in some neighborhood of form (6) (see Goursat, 1933, v. 1, Ch. IX, §§ 181–188). Frequently a transformation of form (4) is given in Taylor series form. Then it is defined in the domain of the convergence of those series. We now return to the problem of level curves.

Theorem 1. *Let the point $X = 0$ be a simple point of the analytic function $f(X)$; i.e., at the origin*

$$\left|\frac{\partial f}{\partial x_1}\right| + \left|\frac{\partial f}{\partial x_2}\right| \neq 0 \ . \tag{8}$$

Then there exists an invertible, analytic coordinate transformation (4) *such that*

$$f(X) = y_2 \ . \tag{9}$$

Proof: Because of condition (8), we know that one of the partial derivatives $\partial f/\partial x_1$ and $\partial f/\partial x_2$ must be non-zero at $X = 0$. Assume that $\partial f/\partial x_2 \neq 0$. We make the coordinate transformation

$$y_1 = x_1 \ , \qquad y_2 = f(X) \ . \tag{10}$$

At $X = 0$, the Jacobian of this transformation is non-zero, since

$$\left(\frac{D(X)}{D(Y)}\right)^{-1} = \det\left(\frac{\partial y_i}{\partial x_j}\right) = \begin{vmatrix} 1 & 0 \\ \dfrac{\partial f}{\partial x_1} & \dfrac{\partial f}{\partial x_2} \end{vmatrix} = \frac{\partial f}{\partial x_2} \neq 0 \ .$$

Obviously, transformation (10) includes expression (9). □

The point of this theorem is that, with the help of a local change of variables, the function f can be written in a simple form which makes its level curves much easier to find. Indeed, the level curves of the function in (9) are, in Y coordinates, just the horizontal lines $y_1 = \tau$, $y_2 = c$. Near the origin of the original coordinate system X, this picture changes a little, and the level curves are no longer straight lines, but as before they will not cross themselves. They will be expressed by the formulas

$$x_i = \xi_i(\tau, c) \ , \qquad i = 1, 2 \ ,$$

where $|\tau| < \varepsilon$, $|c| < \varepsilon$. In fact, the same result is obtained by solving the equation $f(x_1, x_2) - c = 0$ with $x_2 = \xi_2(x_1, c)$.

3.3. A Nondegenerate Critical Point

In what follows, we will consider only a neighborhood of the critical point $X = 0$, when expansion (3) contains no linear terms. If the discriminant of the quadratic terms, $d = f_{(1,1)}^2 - 4f_{(2,0)}f_{(0,2)}$, is non-zero, then the critical point $X = 0$ is *nondegenerate*; if $d = 0$, the point is *degenerate*.

As we know from elementary algebra, every quadratic form

$$f_{20}x_1^2 + f_{11}x_1x_2 + f_{02}x_2^2 \not\equiv 0$$

can, with the help of a non-singular linear transformation $X = BZ$, be transformed into a sum of quadratic terms

$$a(z_1^2 + \sigma z_2^2) \ , \qquad a \neq 0 \ ,$$

where the number σ is zero or ± 1. For a non-degenerate critical point, of course,

$\sigma \neq 0$. The expansion of f in terms of Z takes the form

$$f = az_1^2 + a\sigma z_2^2 + \sum_{q_1+q_2 \geqslant 3} \tilde{f}_Q Z^Q . \tag{11}$$

Theorem 2. *Let $X = 0$ be a non-degenerate critical point for a function f. Then there exists an invertible analytic transformation* (4) *which reduces the function f to the form*

$$f = a(y_1^2 + \sigma y_2^2) . \tag{12}$$

Proof. Let the function f be reduced to form (11). As noted, $a \neq 0$ and $\sigma \neq 0$. We divide the last sum in (11) into three parts,

$$h^{(1)}(Z) = \sum_{q_2<2} \tilde{f}_Q Z^Q ,$$

$$h^{(2)}(Z) = \sum_{q_1<2} \tilde{f}_Q Z^Q ,$$

$$h^{(3)}(Z) = \sum_{q_1, q_2 \geqslant 2} \tilde{f}_Q Z^Q .$$

Thus,

$$h^{(1)} + h^{(2)} + h^{(3)} = \sum_{q_1+q_2 \geqslant 3} \tilde{f}_Q Z^Q .$$

In figure 25, the different shaded regions of the q_1, q_2 plane contain the supports of the different series $h^{(i)}$. Let $g(Z)$ be an arbitrary convergent series in Z, the support $\mathbf{D}(g)$ of which satisfies the inequality $q_1, q_2 \geqslant 2$; i.e., it lies in the doubly hatched region in figure 25. We will show that the system of equations

$$\begin{aligned} az_1^2 + h^{(1)}(Z) + g(Z) &= ay_1^2 , \\ a\sigma z_2^2 + h^{(2)}(Z) + h^{(3)}(Z) - g(Z) &= a\sigma y_2^2 \end{aligned} \tag{13}$$

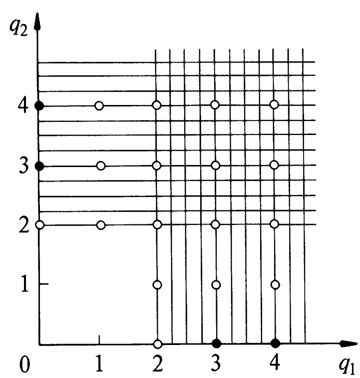

Fig. 25

has an analytic solution. In fact, the first equation immediately gives

$$y_1 = z_1\sqrt{1 + a^{-1}z_1^{-2}(h^{(1)} + g)}\ . \tag{14}$$

But according to our construction, the ratio $(h^{(1)} + g)/z_1^2$ is analytic at the origin and vanishes there. Therefore, the square root in (14) can be expanded in powers of that ratio and, consequently, can be written as a convergent series in powers of Z. It can similarly be shown that the solution $y_2 = y_2(Z)$ of the second equation in (13) is also analytic. □

Note that this theorem allows for considerable arbitrariness in constructing the transformation that leads to (12) since the choice of $g(Z)$ is arbitrary. For example, we could choose $g(Z) = 0$ or $g(Z) = h^{(3)}$. Moreover, there is even greater arbitrariness in the choice of the transformation of theorem 2.

We now investigate the level curves of function (12) in Y coordinates. If $\sigma > 0$, the level curves are the circles

$$y_1^2 + y_2^2 = a^{-1}c = \text{const}$$

(figure 26); f has an extremum at the origin. This last equation can be written

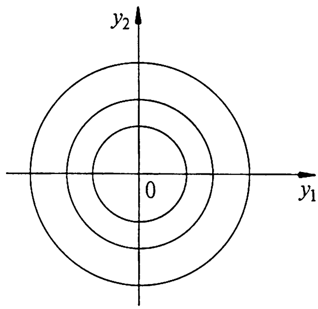

Fig. 26

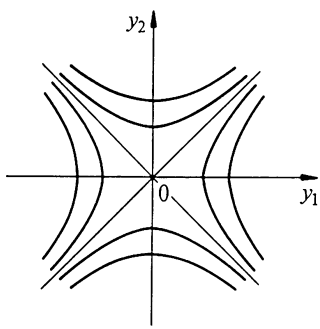

Fig. 27

as a pair of parametric equatons

$$y_1 = \sqrt{c/a}\cos\tau \ , \qquad y_2 = \sqrt{c/a}\sin\tau \ , \qquad ac > 0 \ , \qquad -\infty < \tau < +\infty \ .$$

If $\sigma < 0$, then the level curves are the hyperbolas, $y_1^2 - y_2^2 = c/a = \text{const}$ (figure 27); f has a saddle point at the origin. The curves are then defined in terms of the hyperbolic functions:

$$ac > 0 \ , \qquad y_1 = \pm\sqrt{c/a}\cosh\tau \ , \qquad y_2 = \sqrt{c/a}\sinh\tau \ ;$$

$$c = 0 \ , \qquad y_2 = \pm y_1 \ ; \qquad -\infty < \tau < +\infty \ .$$

$$ac < 0 \ , \qquad y_1 = \sqrt{-c/a}\sinh\tau \ , \qquad y_2 = \pm\sqrt{-c/a}\cosh\tau \ .$$

Here, the parametric equations for the level curves are different in different parts of the neighborhood of $Y = 0$. We obtain an explicit representation (2) of the level curves by substituting into (4) the expressions for y_i in terms of c and τ.

Theorem 2 tells us that in the class of analytic changes of coordinates a function f can be set equal to its own quadratic form if the discriminant of the quadratic form is non-zero.

Example 1. Let us find the level curves of the function

$$f = x_1^3 + x_2^3 - 2x_1x_2 \ .$$

The discriminant is $d = 4 \neq 0$. According to theorem 2, $f = -2y_1y_2$. Consequently, the point $x_1 = x_2 = 0$ is a saddle point of f. The branches of the folium of Descartes pass through this point (see example 5 of section 2.9).

3.4. Sets of Convergence of Powers Series of Class $\mathscr{V}$

In order to prepare a means for studying the neighborhood of a degenerate critical point, we consider a few properties of power series in two variables.

Following Goursat [1933, §§ 161–164], we will consider the series

$$\sum_{Q \in \mathbf{Z}^2} a_Q \ , \tag{15}$$

where the indices $Q = (q_1, q_2)$ run over the two-dimensional integral lattice $\mathbf{Z}^2$. Let these indices Q be represented as points of the $\mathbf{R}_1^2$ plane. Let us imagine that there is an infinitely expanding sequence of regions in this plane

$$\Omega_1 \subset \Omega_2 \subset \Omega_3 \subset \cdots \subset \Omega_n \subset \cdots \ ,$$

which tends to $\mathbf{R}_1^2$ as $n \to \infty$. Let

$$s_n = \sum_{Q \in \Omega_n} a_Q$$

be the sum of those terms of series (15) whose indices lie within the region Ω_n. If the partial sum s_n approaches some limit s as n increases without bound, and if that limit does not depend on the choice of the sequence of regions $\{\Omega_n\}$, then the series (15) *converges to the sum s*.

If, for some sequence of regions $\{\Omega_n\}$, the partial sums of the series

$$\sum |a_Q| \ , \tag{16}$$

approach some limit, then the series (15) is *absolutely convergent*. In this case the sums of series (15) and series (16) do not depend on the choice of the sequence of regions $\{\Omega_n\}$.

Everything we have asserted applies as well to the summation of a multiple power series of the form

$$h = \sum h_Q X^Q \ , \tag{17}$$

where $Q = (q_1, q_2)$, $X^Q = x_1^{q_1} x_2^{q_2}$, and $Q \in \mathbf{Z}^2$. Here, however, a question arises: for what values of X does such a series converge?

Let $\mathbf{V}$ be a closed cone in the $\mathbf{R}_1^2$ plane, the boundary rays of which are spanned by the vectors R_* and R^* (figure 28). Let $\mathscr{V}$ denote the class of power series of form (17) whose supports lie in the cone $\mathbf{V}$:

$$\mathbf{D}(h) \subset \mathbf{V} \ . \tag{17$'$}$$

We assume at first that the coefficients h_Q of such a series are non-negative, and that the variables x_1 and x_2 also take on non-negative values. Let two points, $\underset{\circ}{X}$ and $\mathring{X}$, satisfy the inequalities

$$\underset{\circ}{X}^{R_*} \leqslant \mathring{X}^{R_*} \ , \qquad \underset{\circ}{X}^{R^*} \leqslant \mathring{X}^{R^*} \ . \tag{18}$$

Since any $Q \in \mathbf{V}$ can be expressed as

$$Q = \alpha R_* + \beta R^* \ ,$$

where $\alpha, \beta \geqslant 0$, the inequalities (18) lead to the result

$$\underset{\circ}{X}^Q = (\underset{\circ}{X}^{R_*})^\alpha (\underset{\circ}{X}^{R^*})^\beta \leqslant (\mathring{X}^{R_*})^\alpha (\mathring{X}^{R^*})^\beta = \mathring{X}^Q \tag{19}$$

Clearly, if series (17) converges at a point $\mathring{X}$, it must likewise converge at any point $\underset{\circ}{X}$ which satisfies the inequalities in (18); in fact,

$$h(\underset{\circ}{X}) \leqslant h(\mathring{X}) \ . \tag{20}$$

Conversely, if the series diverges at a point $\underset{\circ}{X}$, it must diverge at any point $\mathring{X}$ which satisfies (18).

In order to analyze the construction of sets of points satisfying (18), we look at the logarithm of the coordinates X:

$$\ln X = (\ln x_1, \ln x_2) \ .$$

Then the inequalities of (18) take the form

$$\langle R_*, \ln \underset{\circ}{X} \rangle \leqslant \langle R_*, \ln \mathring{X} \rangle \ , \qquad \langle R^*, \ln \underset{\circ}{X} \rangle \leqslant \langle R^*, \ln \mathring{X} \rangle \ . \tag{21}$$

Note that the transformation

$$P = \ln X$$

maps the interior of the first quadrant of the X plane onto the P plane. The mapping is monotonic and one-to-one. This transformation maps the point $x_1 = x_2 = 1$ to the origin, $p_1 = p_2 = 0$; the positive half of the x_1 axis is transformed into the "infinitely distant line" $p_2 = -\infty$; the point $x_1 = x_2 = 0$ becomes the "infinitely distant point" $p_1 = p_2 = -\infty$, and so forth.

Let $\tilde{P}$ be some point in the $\mathbf{R}_2^2$ plane. We pass through it two lines which are perpendicular to the vectors R_* and R^*:

$$\begin{aligned} \mathbf{K}_* &= \{P \colon \langle R_*, \tilde{P} \rangle = \langle R_*, P \rangle\} \ , \\ \mathbf{K}^* &= \{P \colon \langle R^*, \tilde{P} \rangle = \langle R^*, P \rangle\} \ . \end{aligned} \tag{21'}$$

These lines divide the plane into four parts (figure 29). The part làbelled III is the normal cone $\mathbf{U}$ of the set $\mathbf{V}$, displaced parallel to the vector $\tilde{P}$. If $\tilde{P} = \ln \underset{\circ}{X}$, then the inequalities in (21) will be satisfied by those $P = \ln \mathring{X}$ which lie in part I. If $\tilde{P} = \ln \mathring{X}$, the inequalities (21) will be satisfied by the points $P = \ln \underset{\circ}{X}$ lying in part III of the plane. Returning to the X plane, we obtain a corresponding division of the first quadrant into four parts (figure 30). If $\tilde{X} = \underset{\circ}{X}$, then the points $\mathring{X}$ from part I satisfy the inequalities (18); if $\tilde{X} = \mathring{X}$, the points $\underset{\circ}{X}$ from part III satisfy (18).

Example 2. $R_* = (1,0)$, $R^* = (0,1)$; then (17) is an ordinary power series. The cone $\mathbf{V}$ is the first quadrant, and $\mathbf{U}$ is the third quadrant. The inequalities of (18) take the form $|\underset{\circ}{x}_i| \leqslant |\mathring{x}_i|$, $i = 1, 2$.

Example 3. $R_* = (2, -1)$, $R^* = (-1, 1)$ (figure 28). For $\tilde{P} = (-1, -1)$, the division of the $\mathbf{R}_2^2$ plane by the lines (21′) is illustrated in figure 29. The corre-

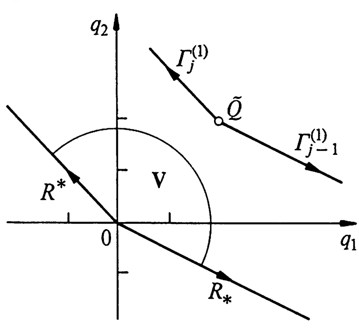

Fig. 28

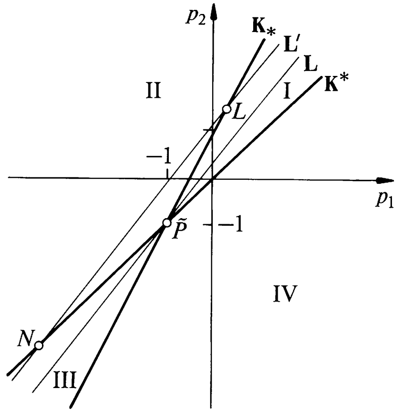

Fig. 29

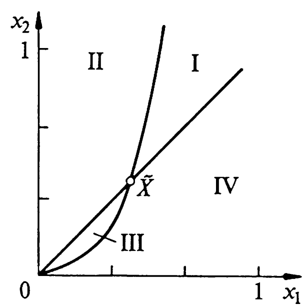

Fig. 30

sponding division of the first quadrant of the x_1, x_2 plane is made by the curves $x_2 = x_1^2$ and $x_2 = x_1$ (figure 30).

Exercise 1. Sketch the cones **V** and the divisions of the X and P planes corresponding to

$$R_* = (2,1), (2,-1), (-1,-2) \ ;$$

$$R^* = (1,2), (-1,2), (-2,-1) \ ;$$

$$\tilde{P} = (-1,0), (-1,1), (0,1), (1,1) \ .$$

Now let us consider a line **L** in the $\mathbf{R}_2^2$ plane, the normal to which does not intersect the cone **V**. If we construct a division of the $\mathbf{R}_2^2$ plane corresponding to a point $\tilde{P} \in \mathbf{L}$, then the line **L** will lie in the resulting parts I and III. We denote

by $\underset{\circ}{\mathbf{L}}(\tilde{P})$ the half-line lying in part III, and by $\mathring{\mathbf{L}}(\tilde{P})$ the half-line lying in part I. The point $\tilde{P}$ is not included in those half-lines. If series (17) converges at the point $\tilde{P} = \ln \mathring{X}$, then it converges for all $\ln X$ lying on the half-line $\underset{\circ}{\mathbf{L}}(\tilde{P})$; likewise, if the series diverges at the point $\tilde{P} = \ln \underset{\circ}{X}$, then it diverges for all $\ln X \in \mathring{\mathbf{L}}(\tilde{P})$. Consequently, there exists some point $M = \ln \tilde{X}$ which lies on the line $\mathbf{L}$, such that series (17) diverges for all $\ln X \in \mathring{\mathbf{L}}(M)$ and converges for all $\ln X \in \underset{\circ}{\mathbf{L}}(M)$. For $\ln X = M$, series (17) may either converge or diverge. An extreme case is possible: M may be one of the "infinite endpoints" of $\mathbf{L}$; then the series either converges for all X, or diverges for all X not on the coordinate axes.

But let us return to the case in which the point M is found in some finite part of the plane. Consider lines parallel to the line $\mathbf{L}$ in the $\mathbf{R}_2^2$ plane. On each of these lines, we can find a point analogous to M. The collection of such points forms a continuous curve in the $\mathbf{R}_2^2$ plane. In fact, let $\mathbf{L}'$ be a second line parallel to $\mathbf{L}$, and let $M' \in \mathbf{L}'$ divide the line $\mathbf{L}'$ into sets of convergence and divergence of series (17). Then M' lies in either the second or fourth parts of the division of the plane corresponding to the point $\tilde{P} = M$; i.e., between the points L and N in figure 29. As the line $\mathbf{L}'$ approaches $\mathbf{L}$, this interval NL converges to the point $M = \tilde{P}$. Hence the curve $\partial\mathbf{M}$, which bounds $\mathbf{M}$, the region of convergence of the series, must be continuous.

Thus, series (17) converges for every point $\ln X = P$ which lies in the region $\mathbf{M}$. At any point outside of $\mathbf{M}$ and $\partial\mathbf{M}$, the series diverges, while at any points of $\partial\mathbf{M}$ it may either converge or diverge.

Depending on series (17), the boundary curve $\partial\mathbf{M}$ may have a variety of forms. If we construct a division of the plane into four parts (as described above) at any point $M \in \partial\mathbf{M}$, the curve $\partial\mathbf{M}$ will lie in the second and fourth parts. Representing the curve $\partial\mathbf{M}$ in the coordinates $x_1 = \exp p_1$ and $x_2 = \exp p_2$, we will obtain a curve $\partial\mathscr{M}_{\mathrm{I}}$ in the first quadrant of the X plane which bounds the region of convergence, $\mathscr{M}_{\mathrm{I}}$, there. This curve will lie in parts II and IV of the division of the plane corresponding to any one of its points.

We now consider in class $\mathscr{V}$ a series (17) with arbitrary numerical coefficients h_Q. We let

$$\overline{|h|} = \sum |h_Q||X|^Q \ .$$

For this series, we construct the curve $\partial\mathscr{M}_{\mathrm{I}}$, which bounds the region $\mathscr{M}_{\mathrm{I}}$ of convergence of the series in the first quadrant of the X plane. Series (17) converges absolutely at a point X if that point lies within the set $\mathscr{M}$ which is bounded by the four curves $\partial\mathscr{M}_{\mathrm{I}}$, $\partial\mathscr{M}_{\mathrm{II}}$, $\partial\mathscr{M}_{\mathrm{III}}$ and $\partial\mathscr{M}_{\mathrm{IV}}$, where the latter three are symmetric to the first with respect to the coordinate axes. We can also show, as does Goursat [1933, § 182], that the series is not absolutely convergent at any point X which lies outside of $\mathscr{M}$ and away from the coordinate axes. The set $\mathscr{M}$ in the X plane may consist of one, two, or four simply connected pieces. If it is not empty, it contains a set

$$|X|^{R_*} \leqslant a \ , \qquad |X|^{R^*} \leqslant b \ . \tag{22}$$

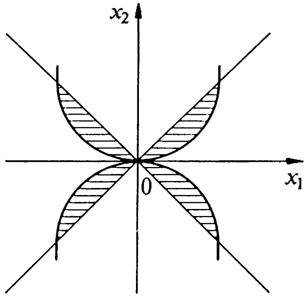

Fig. 31

The set is illustrated for the situation in example 3 as the shaded region in figure 31. We will call a series (17) of class $\mathscr{V}$ *convergent* if its region of absolute convergence contains a set of form (22); otherwise, we shall call it *divergent*.

3.5. Properties of Class $\mathscr{V}$ Series

We consider a convergent series (17) of class $\mathscr{V}$ (i.e., having property (17′)) in its region $\mathscr{M}$ of absolute convergence.

Theorem 3. *The sum h of a series* (17) *of class* $\mathscr{V}$ *is an analytic function at all interior points of the region* $\mathscr{M}$, *with the possible exception of the coordinate axes.*

Proof. If the cone $\mathbf{V}$ lies entirely in the first quadrant of the $\mathbf{R}_1^2$ plane, then the sum h of series (17) is analytic at the point $X = 0$ and at all interior points of convergence (see Goursat, 1933). If $\mathbf{V}$ does not lie in the first quadrant then, according to lemma 1 below, there exists a unimodular matrix α such that the transformation

$$\ln Z = \alpha \ln X \tag{23}$$

transforms series (17) into

$$h' = \sum h_Q Z^{Q'} , \qquad Q' = \alpha^{*-1} Q , \tag{24}$$

a series whose support $\mathbf{D}' = \alpha^{*-1}\mathbf{D}$ lies in the first quadrant of the q_1', q_2' plane.

In the $\mathbf{R}_2^2$ plane, with coordinates $\ln|z_1|$, $\ln|z_2|$, the convergence set $\mathbf{M}'$ of series (24) is related to the convergence set $\mathbf{M}$ of series (17) by the affine transformation: $\mathbf{M}' = \alpha\mathbf{M}$. But the sum h' must be analytic in the set $\mathbf{M}'$. Transformation (23) is analytic everywhere except, perhaps, on the coordinate axes.

Therefore the sum of series (17) will be analytic everywhere within the set of convergence $\mathscr{M}$, with the possible exception of the coordinate axes. □

Lemma 1. *For every convex cone* $\mathbf{V} \subset \mathbf{R}_1^2$ *which does not contain a half-plane, there exists a unimodular matrix* α *such that the cone* $\alpha^{-1*}\mathbf{V}$ *lies in the first quadrant.*

Proof. Let the boundary ray $\mathbf{R}_* = \{cR_*\}$ of the cone $\mathbf{V}$ pass through integral points. Let $Q_* = (q_{1*}, q_{2*})$ be the point nearest to the origin. Let $\sigma = \operatorname{sgn}(q_{2*}r_1^* - q_{1*}r_2^*)$; we seek a pair of integers $\tilde{q}_1, \tilde{q}_2$ which solves the equation

$$q_{2*}\tilde{q}_1 - q_{1*}\tilde{q}_2 = \sigma .$$

If the point $\tilde{Q} = (\tilde{q}_1, \tilde{q}_2)$ lies inside the cone $\mathbf{V}$, then we choose a positive integer k such that the point $Q^* = (q_1^*, q_2^*) = \tilde{Q} - kQ_*$ lies outside of $\mathbf{V}$. This is possible since the cone $\mathbf{V}$ does not contain a half-plane. The cone $\tilde{\mathbf{V}}$, spanned by the vectors Q_* and Q^*, contains the cone $\mathbf{V}$. The matrix

$$\alpha = \begin{pmatrix} q_{1*} & q_{2*} \\ q_1^* & q_2^* \end{pmatrix}$$

is unimodular, and the transformation

$$\alpha^* Q' = Q$$

maps the cone $\mathbf{V}$ onto the first quadrant; the vectors Q_* and Q^* are mapped onto the unit basis vectors. The cone $\mathbf{V}' = \alpha^{*-1}\mathbf{V}$ lies entirely within the cone $\tilde{\mathbf{V}}'$, hence is in the first quadrant.

If neither of the boundary rays of the cone $\mathbf{V}$ is rational (i.e., passes through an integral point), then, instead of $\mathbf{V}$, we must use a larger cone $\mathbf{W}$, which is bounded by a rational ray but which does not contain a half-plane. This is possible because the rational numbers are everywhere dense among the reals. The rest of the construction proceeds as above. □

It is possible to show, as Goursat did, that series (17) is termwise differentiable within its region of convergence. The region of convergence of the partial derivative of the series coincides with that of the series itself; the operations of summation and differentiation commute in that region.

Thus, in a region of convergence series of class $\mathscr{V}$ have the same properties as Taylor series of analytic functions; one can perform arithmetic operations and differentiation on them, and substitute one series into another.

If the cone $\mathbf{V}$ does not lie in the first quadrant, then the point $X = 0$ is not included in the convergence set $\mathscr{M}$ of series (17). But if X approaches the origin within $\mathscr{M}$, then the sum h approaches a limit h_0. As in the proof of Theorem 3, this can be shown using transformation (23) and Lemma 1.

3.6. Transformations of Class $\mathscr{V}^*$

We denote by $\mathscr{V}^*$ the class of transformations of the form

$$y_i = x_i(1 + g_i(X)) \ , \qquad g_i \in \mathscr{V} \ , \qquad g_i(0) = 0 \ , \qquad i = 1, 2 \ . \tag{25}$$

Such a transformation is defined on the intersection of the sets of convergence of series g_1 and g_2 and contains a set of form (22) (see section 3.4).

Let X range over set (22). We will show that the range of values of Y defined by (25) contains a set

$$|Y|^{R_*} \leqslant a' \ , \qquad |Y|^{R^*} \leqslant b' \ . \tag{26}$$

Indeed, let $\mathscr{U}_{\mathbf{V}}$ be a set of form (22), on which $|g_i| \leqslant 1/2$, $i = 1, 2$. On this set according to (25),

$$\tfrac{1}{2}|x_i| \leqslant |y_i| \leqslant \tfrac{3}{2}|x_i| \ , \qquad i = 1, 2 \ .$$

That is,

$$\tfrac{2}{3}|y_i| \leqslant |x_i| \leqslant 2|y_i| \ , \qquad i = 1, 2 \ ,$$

and

$$\mu_* |Y|^{R_*} \leqslant |X|^{R_*} \leqslant \nu_* |Y|^{R_*} \ ,$$

$$\mu^* |Y|^{R^*} \leqslant |X|^{R^*} \leqslant \nu^* |Y|^{R^*} \ ,$$

where μ_*, μ^*, ν_* and ν^* are positive numbers, products of powers of 2 and of 2/3. Hence, the image of set (22) will contain set (26), where $a' = \nu_*^{-1} a$ and $b' = b/\nu^*$.

Theorem 4. *If transformation* (25) *belongs to the class* $\mathscr{V}^*(X)$ *and converges, then there exists a unique inverse transformation*

$$x_i = y_i(1 + h_i(Y)) \ , \qquad i = 1, 2 \ , \tag{27}$$

which also belongs to the class $\mathscr{V}^*$ *and converges.*

Proof. The proof naturally divides itself into two parts. First we prove the existence of the formal series

$$h_i = \sum h_{iQ} Y^Q \in \mathscr{V} \ , \qquad i = 1, 2 \ , \tag{28}$$

which satisfy the equations

$$y_i = y_i(1 + h_i)(1 + g_i(y_1 + y_1 h_1, y_2 + y_2 h_2)) \ , \qquad i = 1, 2 \ . \tag{29}$$

We will then show that the series converge.

To start out, we assume that the cone $\mathbf{V}$ lies in the first quadrant. We will solve equation (25) by the method of undetermined coefficients. We know that in the expansions

$$g_i = \sum g_{iQ} X^Q \ , \qquad i = 1, 2 \tag{30}$$

$g_{i0} = 0$ and, if Q is not in $\mathbf{V}$, $g_{iQ} = 0$. We need to show that there exist solutions

(28) of the system of equations (29) such that $h_{i0} = 0$ and $h_{iQ} = 0$ if Q is not in **V**. We will prove this by induction with respect to the degree of the terms, $\|Q\| = q_1 + q_2$. Let the result be true for all Q: $q_1 + q_2 < n$; we will show it is true for $q_1 + q_2 = n$. Using expression (30), equation (29) can be written as

$$1 = 1 + h_i + \sum g_{iP} Y^P (1 + h_1)^{p_1} (1 + h_2)^{p_2} (1 + h_i) \ , \qquad i = 1, 2 \ . \tag{31}$$

Thus, the coefficient of $Y^Q(\|Q\| = n)$ in the right-hand side of the i^{th} equation has the form

$$h_{iQ} + \sum g_{iP} h_{1P_1} \ldots h_{1P_k} h_{2S_1} \ldots h_{2S_l} \equiv h_{iQ} + c_{iQ} \ , \tag{32}$$

where

$$P + P_1 + \cdots + P_k + S_1 + \cdots + S_l = Q \ .$$

Here, $\|P_j\| < \|Q\|$, $\|S_j\| < \|Q\|$. If $Q \in \mathbf{V}$, we let $h_{iQ} = -c_{iQ}$ and equation (31) is satisfied "for the Y^Q term". If Q does not lie in the cone, then at least one of the indices P, P_j, or S_j in (32) lies outside of **V**, since any sum of vectors in **V** must itself lie in **V**. Then, from our inductive assumption and by the properties of the series g_i, we see that the corresponding coefficient g_{iP}, h_{1P_j}, or h_{2S_j} must vanish. This causes all the terms of the sum c_{iQ} in (32) to vanish, so that $c_{iQ} = 0$; equation (27) is then satisfied by setting $h_{iQ} = 0$. Thus property (28) is proven.

The convergence and uniqueness of the formal series $h_i(Y)$ follow from a general theorem on analytic implicit functions (see Goursat, 1933, part 2, § 185).

We now drop the assumption that the cone **V** lies in the first quadrant and consider the general case. We make the power transformation

$$\begin{aligned} \ln Z &= \alpha \ln X \ , \\ \ln W &= \alpha \ln Y \ , \end{aligned} \tag{33}$$

where the matrix α is as in Lemma 1. Then transformation (25) becomes

$$w_i = z_i(1 + g_i'(Z)) \ , \qquad i = 1, 2 \ , \tag{34}$$

where $\mathbf{D}(g_i') \subset \mathbf{V}' = \alpha^{-1} {}^* \mathbf{V}$. According to Lemma 1, $\mathbf{V}'$ lies in the first quadrant, and we have already proven the theorem for that case. There exists an inverse transformation to (34),

$$z_i = w_i(1 + h_i'(W)) \ , \qquad i = 1, 2 \ .$$

We now use (33) to return to X and Y coordinates from Z and W coordinates. □

Remark. If the cone **V** does not lie in the first quadrant, it is possible to invert transformation (25) without a power transformation. It is sufficient to partially order the indices Q: a vector S precedes a vector Q if $\langle S, T \rangle > \langle Q, T \rangle$, where T is an arbitrary vector in **U**, the normal cone to **V**.

The convergent transformation (25) of the class $\mathscr{V}^*$ is defined on some set (22). According to theorem 4, the inverse transformation (27) is also convergent and of class $\mathscr{V}^*$. As shown just before theorem 4, the domain of definition of the transformation contains a set

$$|X|^{R_*} \leqslant a' \ , \qquad |X|^{R^*} \leqslant b' \ .$$

Let $a'' = \min[a, a']$, $b'' = \min[b, b']$. Then the original transformation (25) and its inverse (27) are defined and single-valued on the set

$$|X|^{R_*} \leqslant \alpha'' \ , \qquad |X|^{R^*} \leqslant b'' \tag{35}$$

and their composition is the identity mapping of that set. Hence, either transformation can be considered as a change of coordinates in region (35).

3.7. The Vertex Case

We return to the problem of constructing the level curves of an analytic function

$$f = \sum f_Q X^Q \ , \qquad f_0 = 0 \ , \qquad 0 \leqslant Q \in \mathbf{Z}^2 \tag{36}$$

in a neighborhood of the degenerate critical point $X = 0$. Let us consider the Newton polygon $\Gamma(f)$. Let $\Gamma_j^{(0)} = \tilde{Q} = (\tilde{q}_1, \tilde{q}_2)$ be a vertex where the edges $\Gamma_{j-1}^{(1)}$ and $\Gamma_j^{(1)}$ meet. Let $R_j = (r_{1j}, r_{2j})$ be a unit vector along $\Gamma_j^{(1)}$ (i.e., it is the difference between neighboring integral points) and let $P_j = (-r_{2j}, r_{1j})$ be a vector normal to $\Gamma_j^{(1)}$ directed out of the polygon Γ. We introduce the vectors $R_* = -R_{j-1}$, $P_* = -P_{j-1}$, $R^* = R_j$, $P^* = P_j$. Then the normal cone $\mathbf{U}_j^{(0)}$ to the vertex $\Gamma_j^{(0)}$ is defined by the inequalities $\langle R_*, P\rangle < 0$, $\langle R^*, P\rangle < 0$, and is bounded by rays directed along P_* and P^* (figures 28 and 32). We will consider the convex cone $\mathbf{V}$ in the $\mathbf{R}_1^2$ plane defined by $\mathbf{V} = \{Q\colon \langle Q, P_*\rangle \leqslant 0, \langle Q, P^*\rangle \leqslant 0\}$, with its vertex at the origin and its bounding rays parallel to the edges $\Gamma_{j-1}^{(0)}$ and $\Gamma_j^{(0)}$ (figure 28).

Theorem 5. *There exists a convergent transformation* (25) *of class* $\mathscr{V}^*$, *which reduces function* (36) *to the form*

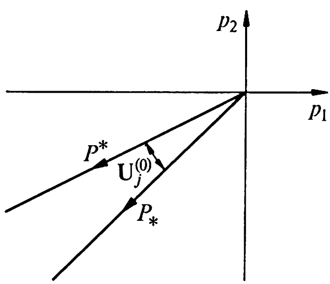

Fig. 32

$$f = f_{\tilde{Q}} Y^{\tilde{Q}} . \tag{37}$$

Proof. So long as $\tilde{Q} \neq 0$, one of the coordinates $\tilde{q}_1$ or $\tilde{q}_2$ is non-zero. For convenience, assume $\tilde{q}_1 \neq 0$. Then the transformation

$$y_1 = x_1 \sqrt[\tilde{q}_1]{f/(f_{\tilde{Q}} X^{\tilde{Q}})} , \qquad y_2 = x_2 \tag{38}$$

reduces function (36) to form (37). The quotient $f/(f_{\tilde{Q}} X^{\tilde{Q}}) = 1 + g(X)$ is a convergent series of class $\mathscr{V}$, with its constant term equal to 1. Therefore the root $\sqrt[\tilde{q}_1]{1 + g(X)}$ can be written as a convergent series in powers of $g(X)$, which yields a class $\mathscr{V}$ series. Consequently, transformation (38) is of class $\mathscr{V}^*$ and is convergent. □

Corollary. *For some* $\varepsilon > 0$, *there exists in the region*

$$\mathscr{U}_j^{(0)}(\varepsilon, X) = \{X : |X|^{R_*} < \varepsilon, |X|^{R^*} < \varepsilon\}$$

a change of coordinates which reduces function (36) *to normal form* (37).

It is sufficient to examine the level curves of the normal form (37) inside some region $\mathscr{U}_j^{(0)}(\varepsilon, Y)$. We write $a = f_{\tilde{Q}}$ and examine a number of different cases.

1) $\tilde{q}_2 = 0$. We can assume that $R_* = (1,0)$. The level curves $a y_1^{\tilde{q}_1} = c$ are vertical lines:

$$y_1 = \begin{cases} \sqrt[q_1]{c/a}, & \text{if } \tilde{q}_1 \text{ is odd, } |c| < \varepsilon ; \\ \pm\sqrt[q_1]{c/a}, & \text{if } \tilde{q}_1 \text{ is even, } 0 < c/a , \end{cases}$$

$$y_2 = \tau, \ |\tau| \leqslant \varepsilon.$$

The dashed curves in figure 33 represent the boundaries of the set $\mathscr{U}_j^{(0)}(\varepsilon, Y)$, while the vertical lines are the level curves. Note that the number c changes sign in $\mathscr{U}_j^{(0)}(\varepsilon, Y)$ only if $\tilde{q}_1$ is odd.

2) $\tilde{Q} > 0$. Here, the level curves $a y_1^{\tilde{q}_1} y_2^{\tilde{q}_2} = c$ are hyperbolas of the form

$$y_1 = \sigma |c/a|^{1/\tilde{q}_1} \tau^{-\tilde{q}_2/\tilde{q}_1} ,$$

$$y_2 = \pm\tau , \qquad 0 < \tau < \infty ,$$

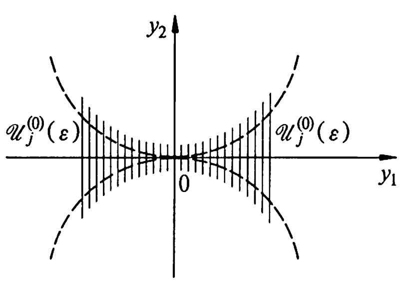

Fig. 33

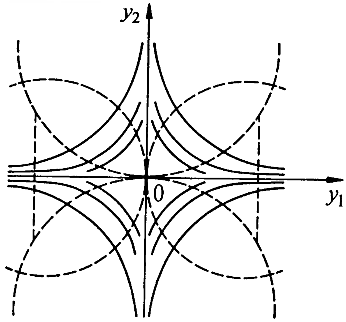

Fig. 34

where σ takes on the values ± 1 depending on the sign of a and c and the parity of $\tilde{q}_1$ and $\tilde{q}_2$. If $c = 0$, the level curves are the coordinate axes (figure 34). If $R_* = (1, 0)$, then the set $\mathcal{U}_j^{(0)}(\varepsilon, Y)$ is just as in figure 33. It contains one line, $y_2 = 0$, which passes through the origin. The other level curves go from one boundary of the set $\mathcal{U}_j^{(0)}(\varepsilon, Y)$ to another.

If the vector R_* lies in the fourth quadrant then the set $\mathcal{U}_j^{(0)}(\varepsilon, Y)$ consists of four parts, each traversed by level curves (figure 34).

Using transformation (37), inverse to (38), we obtain explicit expressions (2) for the level curves in the set $\mathcal{U}_j^{(0)}(\varepsilon, X)$.

Thus, with each vertex $\Gamma_j^{(0)}$ of the Newton open polygon $\hat{\Gamma}(f)$ we can associate a set $\mathcal{U}_j^{(0)}(\varepsilon, X)$, within which the function f reduces to a normal form and is easily studied. In the $\mathbf{R}_2^2$ plane there is a corresponding set

$$\mathrm{U}_j^{(0)}(\varepsilon) = \{P\colon \langle P, R_* \rangle < \ln \varepsilon, \langle P, R^* \rangle < \ln \varepsilon\} ,$$

which results from the translation of the normal cone $\mathbf{U}_j^{(0)}$ along the vector $(\ln \varepsilon, \ln \varepsilon)$.

Exercise 2. Construct the set $\mathcal{U}_j^{(0)}(\varepsilon)$ for the function of example 1, taking $\varepsilon = 1/10$.

3.8. An Edge of the Newton Open Polygon

Now consider $\Gamma_j^{(0)}$ and $\Gamma_{j+1}^{(0)}$, two vertices of the Newton open polygon $\hat{\Gamma}(f)$ (see section 2.9). We will show that the points of the set $\mathcal{U}_{j+1}^{(0)}(\varepsilon)$ are placed above those of the set $\mathcal{U}_j^{(0)}(\varepsilon)$ in the first quadrant of the x_1, x_2 plane. In fact, let $R = R_j$ be the unit vector on the edge $\Gamma_j^{(1)}$. Then the upper boundary of the set $\mathcal{U}_j^{(0)}(\varepsilon)$ is the curve

$$x_1^{r_1} x_2^{r_2} = \varepsilon \ , \quad \text{i.e.} \quad x_2 = \varepsilon^{1/r_2} x_1^{\gamma} \ ,$$

while the lower boundary of the set $\mathscr{U}_{j+1}^{(0)}(\varepsilon)$ is given by

$$x_1^{-r_1} x_2^{-r_2} = \varepsilon \ , \quad \text{i.e.} \quad x_2 = \varepsilon^{-1/r_2} x_1^{\gamma} \ .$$

Since $r_2 > 0$ and $\varepsilon < 1$, then $\varepsilon^{1/r_2} < \varepsilon^{-1/r_2}$, and the second curve is above the first. Consequently, between curvilinear sectors of the sets $\mathscr{U}_j^{(0)}(\varepsilon)$ and $\mathscr{U}_{j+1}^{(0)}(\varepsilon)$ there are other curvilinear sectors of the plane which adjoin the origin. We will associate these sectors with the edge $\Gamma_j^{(1)}$ and denote their union by $\mathscr{U}_j^{(1)}(\varepsilon)$. Thus,

$$\mathscr{U}_j^{(1)}(\varepsilon) = \{X : \varepsilon \leqslant |x_1|^{r_{1j}} |x_2|^{r_{2j}} \leqslant \varepsilon^{-1}, |x_1| \leqslant \varepsilon, |x_2| \leqslant \varepsilon\} \ . \tag{39}$$

This brings us to a new **problem**: find the level curves of f in these sets (39) corresponding to the edges of the Newton open polygon $\hat{\Gamma}(f)$. To solve this problem, we will employ the transformations used in the "second method" of resolution of a singularity (see section 2.9). We note further that, in the $\mathbf{R}_2^2$ plane with coordinates $p_i = \ln|x_i|$, set (39) becomes a half-strip parallel to the vector P. We make a change of coordinates

$$y_1 = x_1^{s_1} x_2^{s_2} \ , \qquad y_2 = x_1^{r_1} x_2^{r_2} \ , \tag{40}$$

where r_1 and r_2 are as above, and s_1 and s_2 are integers chosen to satisfy

$$s_1 r_2 - s_2 r_1 = 1 \ . \tag{41}$$

For any relatively prime numbers r_1 and r_2 there is an infinite number of pairs of integers s_1 and s_2 which satisfy (41) (see section 1.9).

In agreement with the formulas for the power transformation of section 2.6, we have

$$x_1^{q_1} x_2^{q_2} = y_1^{r_2 q_1 - r_1 q_2} y_2^{-s_2 q_1 + s_1 q_2} \equiv y_1^{q_1'} y_2^{q_2'} \ ,$$

that is, the exponent of y_1 is

$$q_1' = -\langle Q', P' \rangle = -\langle Q, P_j \rangle \ ,$$

where $P_j = (-r_2, r_1)$ is a vector orthogonal to the edge $\Gamma_j^{(1)}$, and $P' = \alpha P_j = (-1, 0)$. Therefore, for all $Q' \in \mathbf{D}'$ we will have $q_1' \geqslant r = -\max\langle Q, P_j \rangle$. Moreover, $q_1' = r$ for those terms $f_Q X^Q$ which appear in the truncated series $\hat{f}_j^{(1)}(X)$; i.e.,

$$\hat{f}_j^{(1)}(X) = \hat{f}_j^{(1)'}(Y) = y_1^r \hat{f}_0'(y_2) \ . \tag{42}$$

For all other terms in the expansion

$$f' = \sum f_{Q'} Y^{Q'}$$

$q_1' > r$. Geometrically, this means that under transformation (40), the edge $\Gamma_j^{(1)}$ becomes a vertical edge $\Gamma_j^{(1)'}$, and all the remaining points of Γ' are to the right of this edge. That is,

$$f(X) = f'(Y) = y_1^r f_0'(y_1, y_2) \ , \tag{43}$$

where $r > 0$ and f_0' contains only non-negative powers of y_1.

We now note that after transformation (40), set (39) takes the form

$$\mathscr{U}_j^{(1)'}(\varepsilon) = \{Y: \varepsilon \leqslant |y_2| \leqslant \varepsilon^{-1}, |y_1| \leqslant \varepsilon\} \ . \tag{44}$$

Thus, after transformation (40), our problem has become: find the level curves of function (43) in region (44). This region is the neighborhood of that part of the y_2 axis which contains neither the origin nor the "infinite" ends of the y_2-axis. Now let y_2^0 be some point of set (44) which lies on the y_2 axis. We will try to find the level curves of f in the neighborhood

$$\mathscr{W}(\varepsilon, y_2^0) = \{Y: |y_1| \leqslant \varepsilon, |y_2 - y_2^0| \leqslant \varepsilon\} \ . \tag{45}$$

To do so, we make the parallel translation $y_2 = y_2^0 + z_2$ and consider the function of y_1 and z_2

$$f(X) = y_1^r f_0'(y_1, y_2^0 + z_2)$$

for three different cases.

1) $\hat{f}_0'(y_2^0) = a \neq 0$. Then the transformation

$$z_1 = y_1 \sqrt[r]{f_0'/a} \ , \qquad z_2 = z_2$$

is analytic and invertible in some neighborhood of the point $y_1 = 0$, $y_2 = y_2^0$, and transforms the function f to the form $f = az_1^r$. The level curves of this function are parallel to the z_2 axis (see case 1 at the end of the previous section).

2) y_2^0 is a simple root of the equation $\hat{f}_0'(y_2) = 0$; i.e., $d\hat{f}_0'/dy_2 = a \neq 0$ at $y_2 = y_2^0$. Then the transformation $w_1 = y_1$, $w_2 = f_0'/a = z_2 + \cdots$ is analytic and invertible in some neighborhood of the point $y_1 = 0$, $y_2 = y_2^0$; it reduces the function f to the form

$$f = aw_1^r w_2 \ .$$

The level curves of this function consist of the horizontal line $w_2 = 0$ and the hyperbolas $w_2 = cw_1^{-r}$ (see case 2 in the previous section).

3) y_2^0 is a multiple root of the equation $\hat{f}_0(y_2) = 0$. Then the point $y_1 = 0$, $y_2 = y_2^0$ is a degenerate critical point of $f'(Y)$. In order to study the level curves of this function in the corresponding neighborhood (45), we must construct the Newton open polygon for the function $g(y_1, z_2) = f'(y_1, y_2^0 + z_2)$, divide neighborhood (45) into sets $\mathscr{W}_j^{(d)}(\varepsilon)$, and so forth.

The union of the sets (45) covers the set (44). Let us take some finite covering $\mathscr{W}(\varepsilon, y_2^{(1)}), \ldots, \mathscr{W}(\varepsilon, y_2^{(m)})$. It contains neighborhoods of all the roots of the equation $f_0'(y_2) = 0$ and a number of neighborhoods of simple points of the function $f_0'(Y)$. Set (44) is a horizontal half-strip in the $\mathbf{R}_2^2$ plane with coordinates $p_1 = \ln|y_1|$, $p_2 = \ln|y_2|$. The sets (45) correspond to similar horizontal, but narrower half-strips in that plane. Altogether, they overlap the half-strip of set (44) twice—once for $y_2 > 0$ and once for $y_2 < 0$. Set (44) is covered by a finite number of the sets

$\mathscr{W}(\varepsilon, y_2^{(j)})$, in which the problem is either immediately solvable, or else can be solved by continuing to reduce the order of the singularity. As shown in section 2.9, the "second method" can completely reduce the singularity in a finite number of steps; i.e., a finite number of steps will bring us to case 1), to case 2), or to case 3′) when f has multiple factors: $f = z_1^r(z_2 - b(z_1))^k g(Z)$, where $b(z_1)$ and $g(Z)$ are analytic functions at $Z = 0$, $b(0) = 0$, and $g(0) = a \neq 0$. Then the transformation

$$w_1 = \sqrt[r]{g(Z)/a} \ , \qquad w_2 = z_2 - b(z_1)$$

is analytic at the origin and reduces the function to the normal form

$$f = a w_1^r w_2^k \ .$$

The level curves are coordinate axes and hyperbolas (see case 2 at the end of the previous section).

3.9. Synthesis and Examples

Thus, a neighborhood $\mathscr{U}$ of the critical point $X = 0$ is covered by a finite number of sets

$$\mathscr{U}_j^{(0)}(\varepsilon) \ , \qquad \mathscr{W}_j^{(1)}(\varepsilon), \dots \ , \tag{46}$$

within each of which there exists an invertible change of coordinates $X \to W$ which transforms the function f into a simpler, "normal form":

$$f = a w_1^{q_1} w_2^{q_2} \ .$$

Using this normal form of f, it is easy to find the level curves of f in each of the sets (46) and write expressions for them in the W coordinates:

$$w_i = w_i(\tau, c) \ , \qquad i = 1, 2 \ .$$

Returning from W coordinates to X, we can obtain an expression of form (2) for parts of the level curves of f in each of the sets (46). It remains to "sew them together", and we will have found the level curves of f in the entire neighborhood $\mathscr{U}$. A difficulty arises here, however: the transformation from X to W coordinates is accomplished in two steps. First, the singularity is reduced by applying a finite number of birational transformations, each of which is the result of a finite number of power transformations and parallel translations. Then f is transformed into normal form by a final coordinate change. However, this *normalizing transformation* (and its inverse) is given as an infinite series. In a real situation, therefore, it is only possible to calculate the transformations approximately (up to terms of some finite degree). As a result, the expressions $X = X(\tau, c)$ for parts of the level curves in each of the sets (46) are all approximate. To sew together solutions in neighboring sectors of the sets (46) it is necessary to obtain approximate solutions close to the true level curves. In different cases, different

amounts of calculations are required. For example, to get a topological picture of the level curves, it is generally not necessary to make the normalizing transformation—it is only necessary to resolve the singularity. However, if class C^k accuracy is required (i.e., up to derivatives of k^{th} order), then it is necessary to calculate the normalizing transformation up to terms of some finite (though unknown) degree. In the examples below, we will limit ourselves to finding the topological character of the level curves (to avoid getting bogged down in calculations). In order to sew together the parts of the level curves in neighboring sectors of the sets (46) as we try to get this topological picture, we need to know how the level curves behave as we approach the boundaries of the sectors. Together with the behavior of the level curves in the interiors, this will give us a full topological picture in the case when one level curve adjoins a critical point inside one of the sets (46). If there is no such level curve, then each level curve goes from one sector to another and doesn't remain entirely in anyone region. From the single-valuedness of the function f, we know that the level curves must be closed curves (like circles).

Example 4. We will find the behavior of the level curves of the function

$$f = 2x_1^3 - x_1^2 x_2 - 2x_1 x_2^2 + x_2^3 + x_1^4 + x_2^4$$

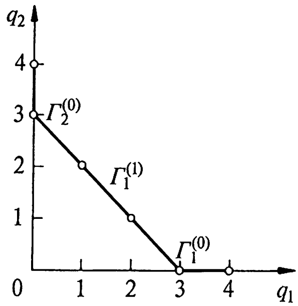

Fig. 35

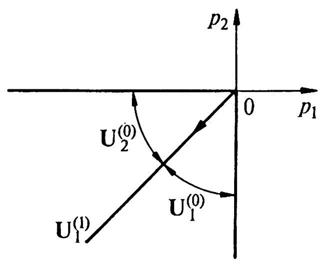

Fig. 36

in a neighborhood of the degenerate critical point $x_1 = x_2 = 0$. The Newton open polygon consists of two vertices, $\Gamma_1^{(0)} = (3,0)$ and $\Gamma_2^{(0)} = (0,3)$, and one edge with unit vector $R = (-1,1)$ (figure 35). Figure 36 shows the normal cones

$$\mathbf{U}_1^{(0)} = \{P: p_1 < 0, p_2 - p_1 < 0\} ,$$

$$\mathbf{U}_2^{(0)} = \{P: p_2 < 0, p_1 - p_2 < 0\} ,$$

$$\mathbf{U}_1^{(1)} = \{P: p_1 = p_2 < 0\} .$$

Let us consider the vertex $\Gamma_1^{(0)}$. The corresponding set $\mathcal{U}_1^{(0)}(\varepsilon)$ in the X plane is

$$\mathcal{U}_1^{(0)}(\varepsilon) = \{X: |x_1| \leqslant \varepsilon, |x_1|^{-1}|x_2| \leqslant \varepsilon\} .$$

In coordinates $p_i = \ln|x_i|$; this set is pictured in figure 37, and in x_1, x_2 coordinates in figure 38. According to theorem 5, there exists a coordinate change $X \to Y$ in this set $\mathcal{U}_1^{(0)}(\varepsilon)$ such that $f = 2y_1^3$; and the coordinate change is close

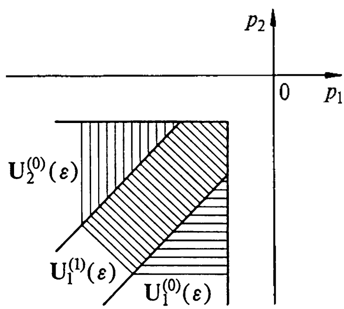

Fig. 37

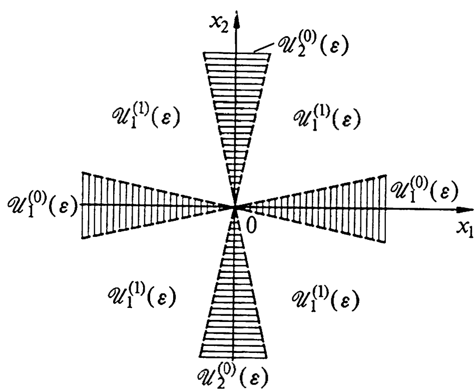

Fig. 38

to the identity near the origin. Consequently, the level curves of f in the set $\mathscr{U}_1^{(0)}(\varepsilon)$ are essentially vertical lines (figure 38).

Similarly, vertex $\Gamma_2^{(0)}$ corresponds to the set

$$\mathscr{U}_2^{(0)}(\varepsilon) = \{X: |x_1||x_2|^{-1} \leqslant \varepsilon, |x_2| \leqslant \varepsilon\}$$

(figures 37 and 38). By theorem 5, there is a change of coordinates $X \to Y$ such that $f = y_2^3$. Thus, the level curves of f in $\mathscr{U}_2^{(0)}$ are essentially horizontal lines (figure 38).

The edge $\Gamma_1^{(1)}$ corresponds to the set

$$\mathscr{U}_1^{(1)}(\varepsilon) = \{X: \varepsilon \leqslant |x_1|^{-1}|x_2| \leqslant \varepsilon^{-1}, |x_i| < \varepsilon\} \tag{47}$$

(figures 37 and 38). In order to investigate the behavior of f in this set, we make the power transformation

$$y_1 = x_1\ , \qquad x_1 = y_1\ ,$$

$$y_2 = x_2 x_1^{-1}\ , \qquad x_2 = y_1 y_2\ .$$

Then

$$f = y_1^3(2 - y_2 - 2y_2^2 + y_2^3 + y_1 + y_1 y_2^4)\ .$$

Here,

$$\mathscr{U}_1^{(1)\prime}(\varepsilon) = \{Y: |y_1| \leqslant \varepsilon \leqslant |y_2| \leqslant \varepsilon^{-1}\}\ . \tag{48}$$

This set consists of the two sets in the y_1, y_2 plane, bounded by dashed lines in figure 39. Here $\hat{f}' = y_1^3 \hat{f}_0'$, where $\hat{f}_0' = 2 - y_2 - 2y_2^2 + y_2^3$. The equation $\hat{f}_0'(y_2) = 0$ has three roots, all of them simple: $y_2 = 1, 2, -1$. The corresponding

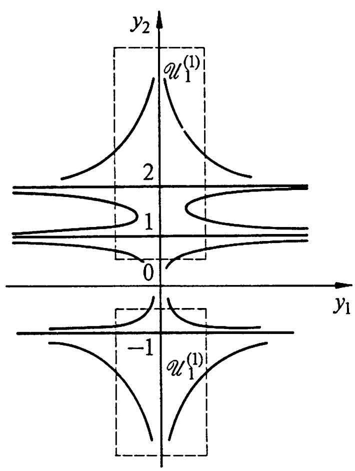

Fig. 39

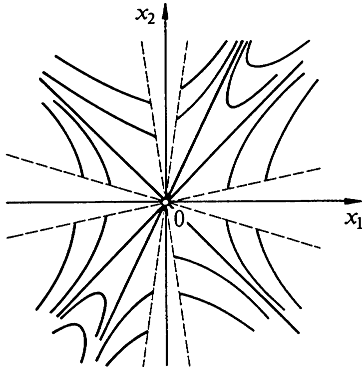

Fig. 40

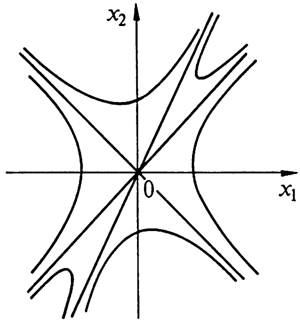

Fig. 41

three points on the y_2 axis are saddle points of f_0'; the other points of that axis are simple points of f_0'. The behavior of the level curves in set (48) is shown in figure 39. Now we could divide set (48) into parts, each part containing no more than one saddle point, and transform f into normal form in each part; but we do not do this. Returning to X coordinates, we find that in set (47), the level curves of f are as illustrated in figure 40. Combining this figure with figure 38, which shows the level curves in the sets $\mathcal{U}_1^{(0)}(\varepsilon)$ and $\mathcal{U}_2^{(0)}(\varepsilon)$, we find that in the entire neighborhood of the origin, the level curves of f behave as illustrated in figure 41.

Example 5. We shall find the behavior of the level curves of the function

$$f = x_2^3 - x_1^3 + x_1^4 + x_2^4 \ .$$

Here, the open polygon $\hat{\Gamma}$, the cones $\mathcal{U}_j^{(d)}$, and the sets $\mathcal{U}_j^{(d)}(\varepsilon)$ are exactly as in

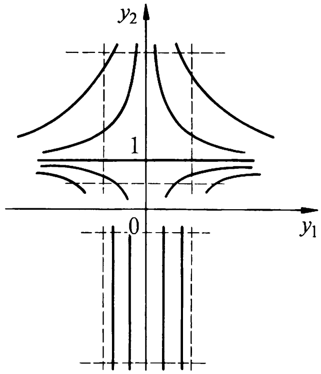

Fig. 42

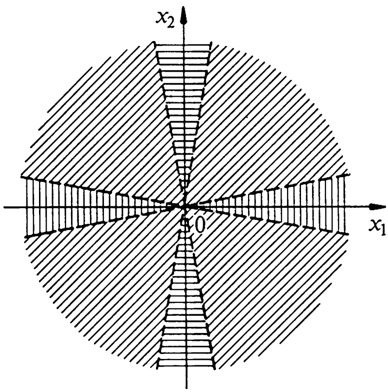

Fig. 43

the preceding example; the level curves behave similarly in the sets $\mathscr{U}_1^{(0)}(\varepsilon)$ and $\mathscr{U}_2^{(0)}(\varepsilon)$. The difference is in the behavior of the curves in the set (48). There, $\hat{f}_0' = y_2^3 - 1$. The equation $\hat{f}_0' = 0$ has three roots, one real ($y_2 = 1$) and two complex. All are simple roots. The behavior of the level curves in set (48) is shown in figure 42. Figure 43 shows the division of the neighborhood of $X = 0$ into eight pieces and the linear approximations of parts of the level curves in each piece. We need only sew together these segments into smooth curves; the result is shown in figure 44.

Exercise 3. Find the behavior (near $X = 0$) of the level curves of the functions f in the exercises and examples of § 2.

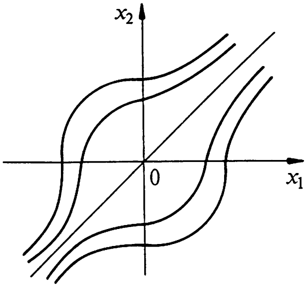

Fig. 44

Remark. In fact, the topological type of the level curves depends entirely on the behavior of the curve $f = 0$. Only branches of this curve pass through the point $x_1 = x_2 = 0$; between two neighbouring branches is a so-called "hyperbolic domain". However, the method presented here allows us to get more complete information about the level curves. For example, the behavior of the level curves in figure 44 is topologically equivalent to that in the neighborhood of a simple point, but there exists no smooth equivalence between these sets of level curves. At the same time, the problem of the level curves of an analytic function serves as an excellent reason for studying the local method. This method can, of course, be applied to more complicated problems, such as investigating the level curves of a ratio of analytic functions $f(X)/g(X)$ near a point $X = X^0$ at which $f(X^0) = g(X^0) = 0$. In Chapter 2 we will use the local method to investigate the integral curves of ordinary differential equations.

3.10. On Normal Forms

The method presented above includes the following. For a given open polygon $\hat{\Gamma}(f)$, we divide a neighborhood $\mathcal{U}$ of the degenerate critical point $X = 0$ into several sets $\mathcal{U}_j^{(d)}(\varepsilon)$, corresponding to the vertices and edges of $\hat{\Gamma}(f)$. Then, in each of the sets $\mathcal{U}_j^{(0)}(\varepsilon)$, we make a transformation to Y coordinates so that

$$f = aY^Q \; . \tag{49}$$

We then divide each of the sets $\mathcal{U}_j^{(1)}(\varepsilon)$ into a finite number of subsets $\mathcal{W}_{jk}(\varepsilon)$, none of which contains more than one root of the truncation $\hat{f}_j^{(1)'}$. If a subset $\mathcal{W}_{jk}(\varepsilon)$ does not contain a multiple root of that truncation, then in it we introduce its own Y coordinates for which (49) is satisfied. Otherwise, we reduce the singularity by constructing a new Newton's polygon and further divide $\mathcal{W}_{jk}(\varepsilon)$ into smaller subsets, just as we did originally for $\mathcal{U}$. After a finite number of steps, this procedure will yield a finite division of the neighborhood $\mathcal{U}$ into subsets $\mathcal{W}_{jk\ldots l}(\varepsilon)$, in each of which the coordinates have been transformed so as to put

f into the simple form of (49). Here, the exponent Q and the coefficient a will be different in different subsets of $\mathcal{U}$. Expression (49) is , naturally, called the "normal form" of f in the subregion $\mathcal{W}_{jk\ldots l}(\varepsilon)$.

If the truncation $\hat{f}_j^{(1)}$ has no multiple roots, we can simplify the function f in the entire set $\mathcal{U}_j^{(1)}(\varepsilon)$ at once:

Theorem 6. *If the truncation $\hat{f}_j^{(1)}(X)$ has no multiple roots away from the coordinate axes, then in the set $\mathcal{U}_j^{(1)}(\varepsilon)$ there exists an invertible coordinate transformation*

$$x_i = y_i(1 + h_i(Y)) \; , \tag{50}$$

which transforms the function f to the form $f = \hat{f}_j^{(1)}(Y)$.

We do not present the proof. We do note that the h_i in transformation (50) are series in integral powers of Y, the supports of which lie in the half plane $V = \{Q\colon \langle Q, P_j \rangle < 0\}$, where $P_j \in \mathbf{U}_j^{(1)}$. Here, V is a *tangent cone.*

Thus, in the set $\mathcal{U}_j^{(1)}(\varepsilon)$, the function f is equivalent to its truncation $\hat{f}_j^{(1)}$. We can therefore say that, in the absence of multiple roots, the truncation is the normal form of the function in the set $\mathcal{U}_j^{(1)}(\varepsilon)$.

Finally, we can limit ourselves to coordinate changes which are invertible and analytic over the entire neighborhood $\mathcal{U}$, and seek the simplest form (the "normal form") to which f can be transformed using such a change of coordinates. It would be appropriate to enumerate at this point a great number of different cases, but there is no general answer (see Arnold, 1975). This is because this class of transformations is extremely limited, and cannot correspond to the many possible complexities in f.

Thus, the form taken by the "normal form" depends on the class of coordinate transformations used. Here, we consider only invertible analytic changes of coordinates which differ depending on the size of the region of definition. The smaller the region, the simpler the normal form.

3.11. On Partitioning Spaces

We now enumerate the objects in various spaces which we have associated with the series

$$f = \sum_{Q \in \mathbf{D}(f)} f_Q X^Q \tag{51}$$

1) In the $\mathbf{R}_1^2$ space with coordinates q_1, q_2, to the series (51) there correspond the set $\mathbf{D}(f)$ (the support) and its convex hull—Newton's polygon $\Gamma(f)$, Newton's open polygon $\hat{\Gamma}$, the edges, $\Gamma_j^{(1)}$, and vertices, $\Gamma_j^{(0)}$, of $\hat{\Gamma}$, and the sets $\mathbf{D}_j^{(d)}$.

2) In the dual space $\mathbf{R}_2^2$ with coordinates p_1, p_2, we have associated series (51) with a partitioning of the third quadrant into cones $\mathbf{U}_j^{(d)}$, corresponding to the edges and vertices of $\hat{\Gamma}$, as well as the sets $\mathbf{U}_j^{(d)}(\varepsilon)$ (for some sufficiently small $\varepsilon > 0$).

3) In the $\mathbf{R}_0^2$ space (with variables x_1 and x_2), we have associated the series (51) with a partitioning of a neighborhood of the point $X = 0$ into sets $\mathcal{U}_j^{(d)}(\varepsilon)$. There is a one-to-one correspondence between these sets and the sets $\mathbf{U}_j^{(d)}(\varepsilon)$. Each of the sets $\mathcal{U}_j^{(d)}(\varepsilon)$ may consist of a number of component pieces which intersect only at $X = 0$. The union of all these sets $\mathcal{U}_j^{(d)}(\varepsilon)$ is the entire neighborhood of $X = 0$. In each of the sets $\mathcal{U}_j^{(d)}(\varepsilon)$, the leading terms of the series (51) (as $X \to 0$) are given by a corresponding truncated series, $\hat{f}_j^{(d)}$. By this we mean the following: Let $\{X^{(n)}, n = 1, 2, 3, \ldots\}$ be a sequence of points of $\mathcal{U}_j^{(d)}(\varepsilon)$ which approach the origin as $n \to \infty$. Then on the sequence $\{X^{(n)}\}$, as $n \to \infty$, all terms $f_Q X^Q$ which are contained in the truncation $\hat{f}_j^{(d)}$, have the same order of magnitude, while the other terms of the series (51) have higher orders of magnitude.

Our entire discussion has been for real variables X. However, all we have said applies equally well to complex-valued x_1 and x_2. The sets $\mathcal{U}_j^{(d)}(\varepsilon)$ will be defined by the same formulas, but will be parts of the two-dimensional complex space $\mathbf{C}_0^2$. The properties described above for truncated series will remain unchanged by a move into complex sets $\mathcal{U}_j^{(d)}(\varepsilon)$. Therefore all our theorems and power-transformation methods (with real matrices α, of course) apply to the case of complex X.

With respect to power transformations, the values $x_i = 0$ and $x_i = \infty$ are equivalent, so the theory we have described can be used to investigate a "neighborhood of infinity". It is then convenient to assume that the space with variables x_1 and x_2 is the Cartesian product of a space $\{x_1\}$ with a space $\{x_2\}$, and that each of the spaces $\{x_i\}$ is either a complex sphere (the complex case) with the single point $x_i = \infty$, or the real part of that sphere also with the single point $x_i = \infty$ (the real case). Such a space, $\{x_1\} \times \{x_2\}$, contains points of the form $x_1 = a, x_2 = \infty$ or $x_1 = \infty, x_2 = \infty$; in the real case, the space is homeomorphic to the two-dimensional torus. The coordinates of the space $\mathbf{R}_2^2$ are $p_i = \ln|x_i|$. It is therefore convenient to consider the space $\mathbf{R}_2^2$ as a Cartesian product of two real lines $-\infty \leqslant p_i \leqslant \infty$, so that it contains points of the form $p_1 = -\infty, p_2 = a$ and $p_1 = -\infty, p_2 = +\infty$, and is homeomorphic to a square.

Those level curves of the polynomial $f(x_1, x_2)$ for which $x_1 \to \infty, x_2 \to \infty$, or $x_1 \to \infty, x_2 \to 0$, can be determined in the same way as those for which $x_1 \to 0$, $x_2 \to 0$. The Newton polygon Γ is constructed, and then that part of its boundary is selected where the normal cone $\mathbf{U}_j^{(d)}$ intersects the *cone* $\mathbf{K}$ *of the problem*. Thus, if $x_1^0 = \infty = x_2^0$, then $\mathbf{K} = \{P: p_1 > 0, p_2 > 0\}$; if $x_1^0 = \infty$, $x_2^0 = \text{const} \neq 0$, then $\mathbf{K} = \{P: p_1 > 0, p_2 = 0\}$; if $x_1^0 = \infty$, $x_2^0 = 0$, then $\mathbf{K} = \{P: p_1 > 0, p_2 < 0\}$, and so on.

Example 6. We shall find the location in the x_1, x_2 plane of the roots of the equation

$$\begin{aligned} f &\equiv x_1^2 x_2^4 + (2x_1 - 6x_1^2)x_2^3 + (x_1^2 - 12x_1 + 1)x_2^2 + (2x_1 - 2)x_2 + 1 \\ &\equiv x_1^2(x_2^4 - 6x_2^3 + x_2^2) + x_1(2x_2^3 + 12x_2^2 + 2x_2) + x_2^2 - 2x_2 + 1 = 0 \ , \end{aligned} \tag{52}$$

and their asymptotic forms for which $x_1 \to 0$ or ∞ as expansions of x_2 in powers of x_1. The set $\mathbf{D}$ (consisting of 9 points) and the Newton polygon Γ (a parallelogram in this case) are shown in figure 44a. The boundary $\partial\Gamma$ contains the four edges $\Gamma_1^{(1)}$. $\Gamma_2^{(1)}$, $\Gamma_3^{(1)}$, $\Gamma_4^{(1)}$ (fig. 44a). In this case

$$\mathbf{U}_1^{(1)} = \{P\colon p_1 < 0, p_2 = 0\} \ , \qquad \mathbf{U}_2^{(1)} = \{P\colon -p_1 = p_2 > 0\} \ ,$$

$$\mathbf{U}_3^{(1)} = \{P\colon p_1 > 0, p_2 = 0\} \ , \qquad \mathbf{U}_4^{(1)} = \{P\colon -p_1 = p_2 < 0\} \ .$$

Hence, the edge $\Gamma_1^{(1)}$ gives a truncation for $x_1 \to 0$, $x_2 \to \text{const} \neq 0$; $\Gamma_2^{(1)}$ gives a truncation for $x_1 \to 0$, $x_2 \to \infty$; $\Gamma_3^{(1)}$ gives a truncation for $x_1 \to \infty$, $x_2 \to \text{const} \neq 0$; $\Gamma_4^{(1)}$ gives a truncation for $x_1 \to \infty$, $x_2 \to 0$. We shall successively treat each of these cases.

$$\textbf{1)}\ \hat{f}_1^{(1)} = x_2^2 - 2x_2 + 1 = (x_2 - 1)^2 \ .$$

Here $x_2 = 1$ is a double root. If we make the substitution $x_2 = 1 + y_2$, then

$$f = y_2^2 - 8x_1 - 16x_1y_2 + O(x_1y_2^2) + O(x_1^2) \ .$$

The Newton open polygon $\hat{\Gamma}$ (the construction of which is left to the reader) consists of a single edge. That is, the unique non-trivial truncation for $x_1 \to 0$, $y_2 \to 0$ is $\hat{f} = y_2^2 - 8x_1$. Its roots are $y_2 = \pm\sqrt{8x_1}$. After the substitution $y_2 = \pm\sqrt{8x_1} + z_2$, we obtain

$$f = \pm 2z_2\sqrt{8x_1} \mp 16x_1\sqrt{8x_1} + O(z_2^2, x_1^2) \ .$$

The terms written out above correspond to the unique non-trivial first approximation for $z_2 = O(\sqrt{x_1})$, as is evident from the corresponding Newton open polygon. Equating the truncation to zero and solving, we obtain the root

$$z_2 = 8x_1 + O(x_1\sqrt{x_1}) \ .$$

Thus, for $x_1 \to 0$, $x_2 \to \text{const}$ there are two roots,

$$x_2 = 1 \pm \sqrt{8x_1} + 8x_1 + O(x_1\sqrt{x_1}) \ ,$$

both of which exist only for $x_1 > 0$.

$$\textbf{2)}\ \hat{f}_2^{(1)} = x_2^2 + 2x_1x_2^3 + x_1^2x_2^4 = x_2^2(1 + x_1x_2)^2 \ .$$

Here there are two double roots, $x_2 = 0$ and $x_1x_2 = -1$. The first of these does not correspond to the cone of truncation (for which $x_2 \to \infty$) and is rejected; the second root $x_2 = -1/x_1$ corresponds to the cone of truncation. After making the power transformation

$$y_1 = x_1 \ , \qquad y_2 = x_1x_2 \ , \tag{53}$$

we obtain

$$\begin{aligned} f &= x_1^{-2}y_2^4 + (2x_1^{-2} - 6x_1^{-1})y_2^3 + (1 - 12x_1^{-1} + x_1^{-2})y_2^2 + (2 - 2x_1^{-1})y_2 + 1 \\ &\equiv x_1^{-2}f_0'(x_1, y_2) \ , \end{aligned} \tag{54}$$

$$f_0' = y_2^4 + 2y_2^3 + y_2^2 - (6y_2^3 + 12y_2^2 + 2y_2)x_1 + (y_2^2 + 2y_2 + 1)x_1^2 \ .$$

To the truncation $\hat{f}_2^{(1)}$ there now corresponds the truncation

$$\hat{f}_0' = y_2^4 + 2y_2^3 + y_2^2 = y_2^2(y_2 + 1)^2 \ .$$

Following the substitution $y_2 = -1 + z_2$, we obtain $f_0'(x_1, y_2) = g(x_1, z_2)$, where

$$g = z_2^2 - 2z_2^3 + z_2^4 - 4x_1 + 4x_1z_2 + 6x_1z_2^2 - 6x_1z_2^2 + z_2^2x_1^2 \ . \tag{55}$$

The roots of the equation $g = 0$ must be found in the neighborhood of the point $x_1 = z_2 = 0$. The Newton open polygon for (55) consists of a single edge, i.e., there is a single non-trivial truncation $\hat{g} = z_2^2 - 4x_1$; its roots are $z_2 = \pm 2\sqrt{x_1}$. We make the substitution $z_2 = \pm 2\sqrt{x_1} + w_2$ and obtain

$$g = w_2^2 \pm 4\sqrt{x_1}w_2 + 40x_1^2 + O(w_2^3, \sqrt{x_1}w_2^2, x_1w_2, x_1^2\sqrt{x_1}) \ .$$

We are interested in the roots of the function g for which $|w_2| = o(\sqrt{x_1})$. From the corresponding Newton open polygon for these roots we obtain the unique truncated equation

$$\pm 4\sqrt{x_1}w_2 + 40x_1^2 = 0 \ .$$

That is, $w_2 = \mp 10x_1\sqrt{x_1} + O(x_1^2)$. Thus, for $x_1 \to 0$, $x_2 \to \infty$ we obtain two roots,

$$x_2 = x_1^{-1}(-1 \pm 2\sqrt{x_1} \mp 10x_1\sqrt{x_1} + O(x_1^2)) \ ,$$

both of which are defined only for $x_1 > 0$.

$$\text{3)}\ \hat{f}_3^{(1)} = x_1^2x_2^2(x_2^2 - 6x_2 + 1) \ .$$

In this case we are only interested in roots for which $x_1 \to \infty, x_2 \to \text{const} \neq 0$, and these are the roots of the equation $x_2^2 - 6x_2 + 1 = 0$. The roots, $x_2 = 3 \pm 2\sqrt{2}$ are simple and positive. After the substitution $x_2 = 3 \pm 2\sqrt{2} + y_2$, we obtain

$$f = \pm 4\sqrt{2}x_1^2y_2 + 12 + O(x_1y_2^3, x_1y_2, y_2) \ . \tag{56}$$

The corresponding Newton polygon is shown in fig. 44b. We are interested only in the roots of the function in (56) for which $x_1 \to \infty$ and $y_2 \to 0$, i.e., the cone of the problem is $\mathbf{K} = \{P: p_1 > 0, p_2 < 0\}$. Of the four edges of the Newton polygon shown in fig. 44b, only the normal cone of edge $\Gamma_1^{(1)}$ enters the cone $\mathbf{K}$. To this edge corresponds the truncation

$$\hat{f} = \pm 4\sqrt{2}x_1^2y_2 + 12 \ ,$$

that is,

$$y_2 = \mp\frac{3}{\sqrt{2}x_1^2}\left[1 + O\left(\frac{1}{x_1}\right)\right].$$

Thus, for $x_1 \to \infty$, $x_2 \to$ const. we have the two roots,

$$x_2 = 3 \pm 2\sqrt{2} \mp \frac{3}{\sqrt{2}x_1^2} + O\left(\frac{1}{x_1^3}\right).$$

Both roots are defined for all values of x_1, other than 0. In this case the larger root does not exceed $3 + 2\sqrt{2}$, while the smaller is no less than $3 - 2\sqrt{2}$.

$$\textbf{4)}\ \hat{f}_4^{(1)} = (x_1 x_2 + 1)^2 .$$

After the power transformation (53) we obtain the polynomial (54), where $\hat{f}_4^{(1)} = (y_2 + 1)^2$. We are interested in the roots for which $x_1 \to \infty$, $y_2 \to$ const. Let $y_2 = -1 + z_2$; then according to (54) and (55) we obtain

$$f = (z_2^2 - 2z_2^3 + 2z_2^4)x_1^{-2} + (-4 + 4z_2 + 6z_2^2 - 6z_2^3)x_1^{-1} + z_2^2 . \tag{57}$$

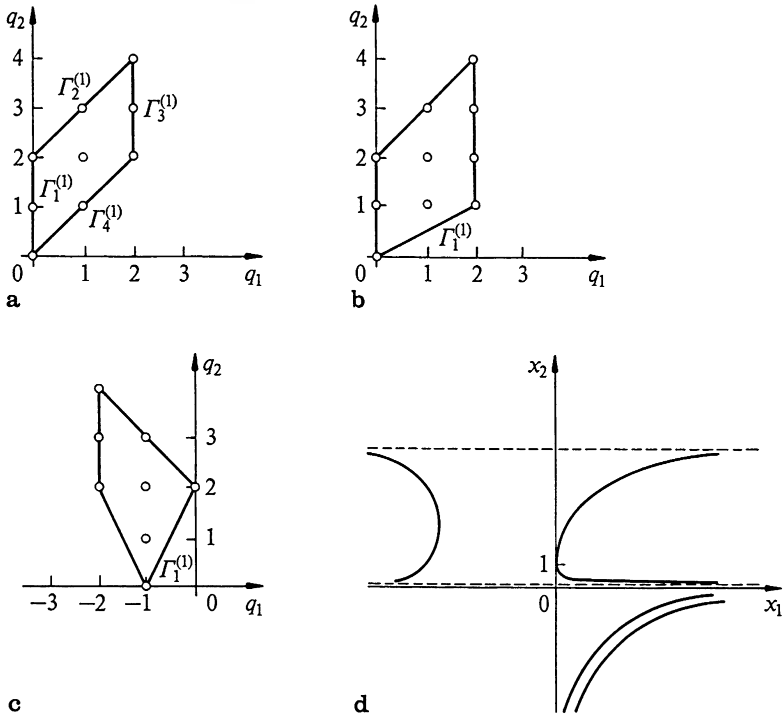

Fig. 44a, b, c, d

The Newton polygon is shown in fig. 44c. Only one of the four edges has its normal cone within the cone of the problem, $\mathbf{K} = \{P: p_1 > 0, p_2 < 0\}$. The corresponding truncation of the polynomial (57) is $-4x_1^{-1} + z_2^2$; that is, $z_2 = \pm 2/\sqrt{x_1} + O(1/x_1)$. Thus, for $x_1 \to \infty$, $x_2 \to 0$ we obtain two roots

$$x_2 = -\frac{1}{x_1} \mp \frac{2}{x_1\sqrt{x_1}} + O\left(\frac{1}{x_1^2}\right).$$

Both are defined only for $x_1 > 0$.

It may be shown that the curve (52) has no critical points; consequently, its branches do not intersect. The branches in the x_1, x_2 plane are shown in fig. 44d (the dashed lines are the asymptotes, $x_2 = 3 \pm 2\sqrt{2}$).

Exercise 4. Find the asymptotes and the disposition on the entire plane of the roots of the polynomials in the examples and exercises of § 2.

Exercise 5. Find the level curves and their graphs of the polynomials in the examples and exercises of § 2.

Remark. The local method in conjunction with a computer may be used to study the singularities and asymptotic properties of the level curves of quite complicated functions which are not polynomials (see Bruno, 1978b).

Chapter II
A System of Two Differential Equations

§ 1. Simple Points and Elementary Singularities

1.1. Introduction

In this chapter, we will consider the solutions of a system of ordinary differential equations

$$\begin{aligned} dx_1/dt &\equiv \dot{x}_1 = \varphi_1(x_1, x_2) \ , \\ dx_2/dt &\equiv \dot{x}_2 = \varphi_2(x_1, x_2) \end{aligned} \tag{1}$$

in some neighborhood of a point $x_1 = x_1^0$, $x_2 = x_2^0$. We will assume that the functions φ_i are analytic in this neighborhood. Our problem is to find the solutions of system (1) in the neighborhood of (x_1^0, x_2^0) in the form $(x_1(\tau), x_2(\tau))$, i.e., as functions of an arbitrary parameter τ.

The character of the t dependence of the variables x_1 and x_2 will not interest us. Therefore it would be more exact to say that we seek the solutions (integral curves) of a single equation

$$\frac{dx_2}{dx_1} = \frac{\varphi_2(x_1, x_2)}{\varphi_1(x_1, x_2)} \ . \tag{2}$$

But since notation (1) is the more convenient of the two, we will use it.

If a solution $x_1 = b_1(\tau)$, $x_2 = b_2(\tau)$ of equation (2) is known, then the first equation of system (1) will tell us that

$$t = \int \frac{db_1(\tau)}{\varphi_1(b_1(\tau), b_2(\tau))} \ ,$$

that is, for a known integral curve, the time dependence of the coordinates can be found with one quadrature.

We make the parallel translation

$$x_1 = x_1^0 + x_1^* \ , \qquad x_2 = x_2^0 + x_2^* \ .$$

Then system (1) becomes

$$\dot{x}_i^* = \varphi_i^*(x_1^*, x_2^*) \equiv \varphi_i(x_1^0 + x_1^*, x_2^0 + x_2^*)\ , \qquad i = 1, 2,$$

which we will investigate in a neighborhood of the origin, $x_1^* = x_2^* = 0$. From now on we will assume that this translation has been made, so that our region of interest is a neighborhood of $x_1 = x_2 = 0$.

If $|\varphi_1| + |\varphi_2| \neq 0$ at the origin, then it is a *simple point* of system (1). If $\varphi_1 = \varphi_2 = 0$ at the origin, then it is a *singular point* of the system; it is a stationary solution of system (1).

At such a singular point, the expansion of the function φ_i begins with terms linear in x_1 and x_2. If we isolate these terms, we can write system (1) in vector form

$$\dot{X} = AX + \cdots, \tag{2'}$$

where A is a square matrix and X is the vector (x_1, x_2). Let λ_1 and λ_2 be the eigenvalues of A (i.e., the roots of the equation $\det(A - \lambda E) = 0$). If $|\lambda_1| + |\lambda_2| \neq 0$, then the origin is an *elementary singular point* of the system; if $\lambda_1 = \lambda_2 = 0$, then the origin is a *non-elementary singular point*. The classification of points for system (1) is similar to that for an analytic function f. The analogy is especially notable in cases in which system (1) has an integral $f = \text{const}$. Unfortunately, different terminology has arisen for the two different situations; we therefore present here the correspondences:

1) simple points of f correspond to simple points of system (1);

2) critical points of f correspond to singular points of system (1); and

3) non-degenerate and degenerate critical points of f correspond, respectively, to elementary and non-elementary singular points of system (1).

There are two points of view in the investigation of the integral curves of system (1) in the neighborhood of a singular point:

1. Seek only those solutions which pass through the singular point. For this, it is sufficient to use the methods employed in § 2 of Chapter I to solve the analytic equation $f(x_1, x_2) = 0$. Works adopting this point of view include those of Briot and Bouquet [1856], Horn [1894], Bendixson [1898], Dulac [1904], and others (see Dulac's review [Dulac, 1934] and § 4 of Bieberbach's book [Bieberbach, 1953]).

2. Seek all integral curves in a neighborhood of the singular point, where the neighborhood may be arbitrarily small. For this approach, one can use the methods employed in § 3 of Chapter I to describe the level curves of an analytic function. This is the point of view adopted in this work.

1.2. The Neighborhood of a Simple Point

In what follows, we shall be making coordinate transformations

$$x_i = \xi_i(y_1, y_2)\ , \qquad \xi_i(0,0) = 0\ , \qquad i = 1, 2\ , \tag{3}$$

in the neighborhood of the origin; these changes of coordinates map the origin onto itself and will be assumed to be invertible (i.e., the Jacobian does not vanish

at the origin). The properties of such changes of coordinates were discussed in section 3.2 of Chapter I.

Theorem 1. *Let the origin be a simple point of system* (1). *There exists an invertible, analytic change of coordinates* (3) *under which system* (1) *becomes*

$$\dot{y}_1 = 1 \ , \qquad \dot{y}_2 = 0 \ . \tag{4}$$

This theorem means that the solutions of system (1) are topologically equivalent to those of system (4) in some sufficiently small neighborhood of a simple point; see figures 45 and 46. We might call this the Cauchy-Arnol'd theorem. Cauchy proved the existence, uniqueness, and analyticity of a solution which passes through a simple point, while Arnol'd [1971b] considered the entire neighborhood of the origin, and formulated and proved Theorem 1. We note that system (4) is very simple; theorem 1 says that the solutions of system (1) in the neighborhood of a simple point have simple structure.

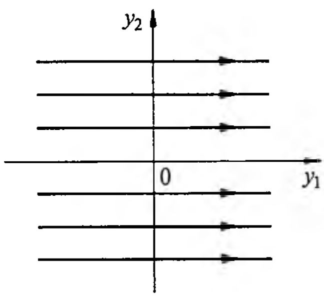

Fig. 45

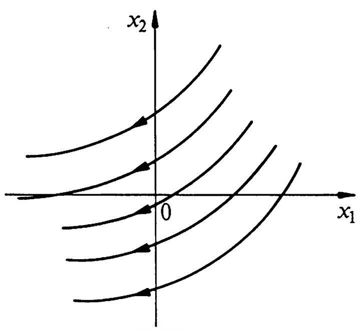

Fig. 46

Proof of Theorem 1. Let us differentiate equation (3) with respect to t:

$$\dot{x}_i = \frac{\partial \xi_i}{\partial y_1}\dot{y}_1 + \frac{\partial \xi_i}{\partial y_2}\dot{y}_2 , \qquad i = 1, 2$$

and substitute the derivatives $\dot{x}_i$ and $\dot{y}_i$ in their expressions in systems (1) and (4). Then we obtain a system of partial differential equations

$$\frac{\partial \xi_i}{\partial y_1} = \varphi_i(X) = \varphi_i(\xi_1(Y), \xi_2(Y)) , \qquad i = 1, 2 . \tag{5}$$

The functions ξ_i must satisfy this system if coordinate change (3) does indeed transform system (1) into system (4). We will show that system (5) has solutions in the form of power series

$$\xi_i = \sum \xi_{iQ} Y^Q ,$$

where $Q = (q_1, q_2)$, $Y = (y_1, y_2)$, $Y^Q = y_1^{q_1} y_2^{q_2}$. Similarly, we can write

$$\varphi_i = \sum \varphi_{iS} X^S .$$

Since

$$X^S = \xi_1^{s_1} \xi_2^{s_2} = (\sum \xi_{1P} Y^P)^{s_1} (\sum \xi_{2R} Y^R)^{s_2} ,$$

then the coefficient of Y^Q in this product will be a sum of terms of the form

$$\xi_{1P_1} \xi_{1P_2} \cdots \xi_{1P_{s_1}} \xi_{2R_1} \xi_{2R_2} \cdots \xi_{2R_{s_2}} ,$$

where

$$P_1 + P_2 + \cdots + P_{s_1} + R_1 + R_2 + \cdots + R_{s_2} = Q . \tag{6}$$

Therefore the coefficient of Y^Q in $\varphi_i(\xi_1, \xi_2)$ is a sum of terms of the form

$$\varphi_{iS} \xi_{1P_1} \cdots \xi_{1P_{s_1}} \xi_{2R_1} \cdots \xi_{2R_{s_2}} , \tag{7}$$

where equation (6) again holds. We define the norm of a vector $Q = (q_1, q_2)$ as $\|Q\| = q_1 + q_2$. Note that $\xi_{iP} \neq 0$ only for $P \geqslant 0$ and $\|P\| \geqslant 1$. As a result, all the P_j and R_j in product (7) will satisfy $\|P_j\| < \|Q\|$ and $\|R_j\| < \|Q\|$ if $\|S\| > 1$ (i.e., if there is more than one term on the left-hand side of equation (6)). If $\|S\| = 1$, then $\|P_j\| = \|Q\|$ or $\|R_j\| = \|Q\|$. But it is always true that

$$\|P_j\| \leqslant \|Q\| , \qquad \|R_j\| \leqslant \|Q\| . \tag{8}$$

That is, the formation of the coefficients of the terms in the series $\varphi_i(\xi_1, \xi_2)$ whose index norms are $\|Q\|$ involves only those coefficients ξ_{1P} and ξ_{2R} of terms in the series $\xi_1(Y)$ and $\xi_2(Y)$ for which $\|P\|$ and $\|R\|$ do not exceed $\|Q\|$.

On the other hand, the coefficient of Y^Q in the series $\partial \xi_i / \partial y_1$ is $(q_1 + 1)\xi_{i(q_1+1, q_2)}$. Equations (5) are only satisfied when the coefficients of Y^Q in the series $\partial \xi_i / \partial y_1$ and φ_i are equal for all Q, i.e., when

$$(q_1 + 1)\xi_{i(q_1+1,q_2)} = \sum \varphi_{iS}\xi_{1P_1}\cdots\xi_{1P_{s_1}}\xi_{2R_1}\cdots\xi_{2R_{s_2}} , \qquad i = 1, 2 . \tag{9}$$

The norm of the vector index of the coefficient $\xi_{i(q_1+1,q_2)}$ is $\|Q\| + 1$; inequalities (8) are satisfied for all terms of form (7). Therefore, if all the coefficients ξ_{iS} with $\|S\| < \|Q\|$ are known, then equation (9) will give us the coefficient $\xi_{i(q_1+1,q_2)}$ for $q_1 + 1 \neq 0$. If $q_1 + 1 = 0$, the left-hand side of equation (9) vanishes, since it contains a zero factor. The right-hand side also vanishes, since the series $\varphi_i(\xi_1, \xi_2)$ do not include negative powers ($q_1 = -1$) of the variables y_1 and y_2. Thus, equation (9) sets no limits on the coefficients $\xi_{i(0,q_2)}$, which can be spcified arbitrarily.

We will begin the determination of the coefficients ξ_{iQ} with the linear terms $\xi_{i(1,0)}y_1 + \xi_{i(0,1)}y_2$, for which equations (5) yield

$$\xi_{i(1,0)} = \varphi_{i(0,0)} , \qquad i = 1, 2 .$$

We will specify the coefficients $\xi_{i(0,1)}$ arbitrarily, giving them values such that the Jacobian does not vanish:

$$\begin{vmatrix} \xi_{1(1,0)} & \xi_{1(0,1)} \\ \xi_{2(1,0)} & \xi_{2(0,1)} \end{vmatrix} \neq 0 . \tag{10}$$

We can do this because one of the coefficients $\varphi_{i(0,0)}$ must be non-zero, since the origin is a simple point of system (1). For example, if $\varphi_{1(0,0)} \neq 0$, and we choose $\xi_{1(0,1)} = 0$ and $\xi_{2(0,1)} \neq 0$, then inequality (10) will be satisfied. This determines the coefficients ξ_{iQ} with $\|Q\| = 1$.

We next find the coefficients of the quadratic terms

$$\xi_{i(2,0)}y_1^2 + \xi_{i(1,1)}y_1y_2 + \xi_{i(0,2)}y_2^2 , \qquad i = 1, 2 .$$

System (5) gives

$$\begin{aligned} 2\xi_{i(2,0)}y_1 + \xi_{i(1,1)}y_2 &= \varphi_{i(1,0)}(\xi_{1(1,0)}y_1 + \xi_{1(0,1)}y_2) \\ &\quad + \varphi_{i(0,1)}(\xi_{2(1,0)}y_1 + \xi_{2(0,1)}y_2) . \end{aligned}$$

This uniquely defines the coefficients $\xi_{i(2,0)}$ and $\xi_{i(1,1)}$, but the coefficients $\xi_{i(0,2)}$ can be arbitrarily specified (as zero, for example). Continuing in this manner, we can find the coefficients ξ_{iQ} of terms of third and higher order. All the coefficients $\xi_{i(0,q_2)}$ may be specified arbitrarily (as zero, for example), and the rest will be uniquely determined.

We can thus find the series $\xi_i(Y)$ which satisfy system (5); with the help of majorizing series (see Petrovskiĭ, 1964) it can be shown that the resulting series ξ_i converge for $|y_1|$, $|y_2| < \varepsilon$, but we will not do so here. The Jacobian of transformation (3) is the non-zero determinant (10), so that the transformation is invertible. The proof is complete. □

We have two comments on this proof. First, transformation (3) was found by the method of undetermined coefficients. These coefficients ξ_{iQ} were somehow ordered (by the magnitude of $q_1 + q_2$, for instance) and were then found successively according to this ordering. Second, some of the coefficients (here, the $\xi_{i(0,q_2)}$) could be arbitrarily specified, while the rest were uniquely determined. We will frequently encounter a similar situation in what follows.

Remark. In theorem 1, the coefficients and variables may be either real or complex. If the original system (1) is real (i.e., if all the coefficients φ_{iQ} are real numbers) then the coefficients ξ_{iQ} will also be real (as long as the $\xi_{i(0,q_2)}$ are taken as real).

1.3. Linear Systems

We consider the system

$$\dot{x}_1 = a_{11}x_1 + a_{12}x_2 \ ,$$

$$\dot{x}_2 = a_{21}x_1 + a_{22}x_2$$

or, in matrix form,

$$\dot{X} = AX \ , \tag{11}$$

where X is a column vector. After the change of coordinates

$$X = BY \ , \qquad \det B \neq 0 \tag{12}$$

system (11) is transformed into

$$\dot{Y} = B^{-1}ABY = JY \ . \tag{13}$$

As is well known (see, for example, Pontryagin, 1961), every system (11) can be transformed, under a coordinate change (12), into a system (13) in which the matrix $J = B^{-1}AB$ is a Jordan matrix. That is,

$$J = \begin{pmatrix} \lambda_1 & 0 \\ \sigma & \lambda_2 \end{pmatrix} ,$$

where λ_1 and λ_2 are the eigenvalues of the matrix A; $\sigma = 0$ or 1 if $\lambda_1 = \lambda_2$, and $\sigma = 0$ if $\lambda_1 \neq \lambda_2$.

In order to find the matrix B which transforms the matrix A into the Jordan matrix J, we first find the eigenvalues of A by solving the characteristic equation $\det(A - \lambda E) = 0$ or

$$\lambda^2 - (a_{11} + a_{22})\lambda + \det A = 0 \ . \tag{14}$$

Then for each of the eigenvalues λ_i we find the associated eigenvector B_i by

solving the system

$$AB_i = \lambda_i B_i \ . \tag{15}$$

This system will determine only the ratio of the elements of the vector B_i (one of the elements must be arbitrarily specified). Once both of the vectors B_i are found, we let them be the columns of the matrix B (we write $B = (B_1, B_2)$). The numbers λ_1, λ_2 and σ are invariants of the system (11) under linear transformations (12).

Example 1. Let $A = \begin{pmatrix} 4 & 3 \\ -1 & 0 \end{pmatrix}$. The characteristic equation (14) is

$$\lambda^2 - 4\lambda + 3 = 0 \ ,$$

whence $\lambda_1 = 3$, $\lambda_2 = 1$. For $B_1 = \begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix}$ we have the system of equations

$$4b_{11} + 3b_{21} = 3b_{11} \ ,$$

$$-b_{11} = 3b_{21} \ .$$

Setting $b_{21} = 1$, we find $b_{11} = -3$. Similarly, for $B_2 = \begin{pmatrix} b_{12} \\ b_{22} \end{pmatrix}$,

$$4b_{12} + 3b_{22} = b_{12} \ ,$$

$$-b_{12} = b_{22} \ ;$$

whence $b_{12} = 1$, $b_{22} = -1$. The matrix B is

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ 1 & -1 \end{pmatrix} .$$

Thus

$$J = B^{-1}AB = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{3}{2} \end{pmatrix} \begin{pmatrix} 4 & 3 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -3 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} .$$

In the event of multiple eigenvalues $\lambda_1 = \lambda_2$, if the matrix A is not diagonal, we find the vector B_2 from system (15) (i.e. B_2 is the unique eigenvector); the vector B_1 is then found from the system

$$AB_1 = \lambda_1 B_1 + B_2 \ . \tag{16}$$

Example 2. Let $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$. The characteristic equation is $\lambda^2 - 2\lambda + 1 = 0$, whence $\lambda_1 = \lambda_2 = 1$. For eigenvector B_2, system (15) is

$$2b_{12} + b_{22} = b_{12} \ ,$$

$$-b_{12} = b_{22} \ ;$$

whence we can write $b_{12} = 1$, $b_{22} = -1$. System (16) then becomes

$$2b_{11} + b_{21} = b_{11} + b_{12} = b_{11} + 1 \ ,$$

$$-b_{11} = b_{21} + b_{22} = b_{21} - 1 \ .$$

This yields $b_{11} = 1$, $b_{21} = 0$. Thus, $B = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$ and

$$J = B^{-1}AB = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} .$$

1.4. An Elementary Singular Point

In this section, we shall discuss what happens when $\varphi_i(0,0) = 0$ in system (1), i.e., when the origin is a singular point of the system. Then, we can write the system (1) in form (2′). Using an invertible linear change of coordinates (12), we can transform the matrix A of the linear terms of system (1) into a normal form (a Jordan matrix), so that the system itself becomes

$$\dot{y}_i = \lambda_i y_i + \sigma_i y_{i-1} + \varphi_i(Y) \ , \qquad i = 1, 2 \ , \tag{17}$$

where $\sigma_1 = 0$, $\sigma_2 = \sigma$, and the series φ_i contain no constant or linear terms. We consider here the case when at least one of the eigenvalues λ_i is non-zero (i.e., $|\lambda_1| + |\lambda_2| \neq 0$); the point $Y = 0$ is then an elementary singular point of system (17).

Our task is to transform system (17) into the simplest possible form using a local, invertible change of coordinates

$$y_i = z_i + \xi_i(z_1, z_2) \ , \qquad i = 1, 2 \ , \tag{18}$$

where the series ξ_i contain neither constant nor linear terms. If we wish transformation (18) to be analytic, then the ξ_i will be power series which converge for sufficiently small $|z_1|$ or $|z_2|$. Our original problem then divides into two:

1) Find the power series ξ_i which transform system (17) into simplest form; i.e., find the coefficients ξ_{iQ} of these series.

2) Investigate the convergence of these series.

In solving the first problem, we will have to deal with power series for which we do not know whether there exists a neighborhood of the point $Z = 0$ in which they converge, or whether they diverge in every neighborhood of the origin. The latter are *formal* power series. Arithmetic operations and differentiation are carried out on them just as on convergent series. Up to now we have met convergent series only, but differential equations yield divergent series as well.

Example 3. The equation

$$x^2 dy/dx = y - x$$

has the power series

$$y = \sum_{k=1}^{\infty} (k-1)! x^k .$$

as its solution. This series diverges for any $x \neq 0$. At the same time, this is the Taylor series of the "true solution"

$$y = e^{-1/x} \int_0^x \frac{e^{1/u}}{u} du ,$$

which is non-analytic but infinitely differentiable at $x = 0$.

We will thus solve the following problem. Suppose (17) is a formal system. The φ_i are formal power series without constant and linear terms, and the matrix of coefficients of linear terms is a Jordan matrix. We will ask, what is the simplest formal system

$$\dot{z}_i = \lambda_i z_i + \sigma_i z_{i-1} + \psi_i(Z) \equiv \tilde{\psi}_i(Z) , \qquad i = 1, 2 \tag{19}$$

to which we can transform system (17) with the help of an invertible formal change of coordinates (18) (where the ξ_i are power series with no constant terms, and the matrix of coefficients of linear terms is the identity matrix)?

To order to formulate an answer, we introduce a new notation:

$$\tilde{\psi}_i = z_i g_i(Z) = z_i \sum_{Q \in \mathbf{N}_i} g_{iQ} Z^Q , \qquad i = 1, 2 , \tag{20}$$

where

$$Q = (q_1, q_2) , \qquad Z = (z_1, z_2) , \qquad Z^Q = z_1^{q_1} z_2^{q_2} .$$

As long as the $\tilde{\psi}_i$ are series in non-negative integral powers of z_1 and z_2 and have no constant terms, the summations of notation (20) will be taken over the sets

$$\mathbf{N}_1 = \{Q\text{: integral } q_1 \geqslant -1, q_2 \geqslant 0, q_1 + q_2 \geqslant 0\} ,$$

$$\mathbf{N}_2 = \{Q\text{: integral } q_1 \geqslant 0, q_2 \geqslant -1, q_1 + q_2 \geqslant 0\} .$$

That is, $\mathbf{N}_i$ is the set of integral points in the q_1, q_2 plane which satisfy the inequalities $q_i \geqslant -1$, $q_j \geqslant 0$ $(j \neq i)$, $q_1 + q_2 \geqslant 0$. The boundaries of the sets satisfying these inequalities are shaded in figure 47.

Exercise 1. Show how the linear terms of system (19) appear in notation (20); that is, express the λ_i and σ_i as g_{iQ}.

1.5. The Principal Theorem on the Normal Form

We introduce notation similar to that of expression (20) for system (17) and transformation (18):

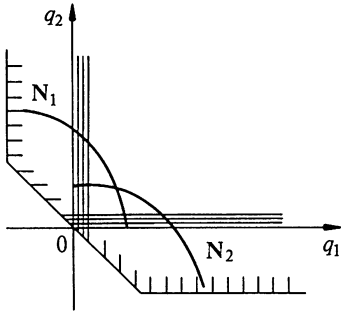

Fig. 47

$$\begin{aligned} \varphi_i &= y_i f_i(Y) = y_i \sum_{Q \in \mathbf{N}_i} f_{iQ} Y^Q , \qquad i = 1, 2 , \\ \xi_i &= z_i h_i(Z) = z_i \sum_{Q \in \mathbf{N}_i} h_{iQ} Z^Q , \qquad i = 1, 2 . \end{aligned} \tag{21}$$

Here, the φ_i, ξ_i, and ψ_i are all formal series in non-negative integral powers of the appropriate variables with no constant and linear terms.

Note that the matrix of the linear terms in transformation (18) is the identity, so that the change of coordinates is invertible; that is, the z_i can be written as power series in y_1 and y_2.

Theorem 2. *For every system* (17), *there exists a change of coordinates* (18) *which transforms it into a system* (19) *for which* (*in notation* (20))

$$g_{i(q_1, q_2)} = 0 , \qquad \text{if} \quad q_1\lambda_1 + q_2\lambda_2 \neq 0 .$$

That is, the only non-zero terms in system (19) will be terms $z_i g_{iQ} Z^Q$ for which

$$\langle Q, \Lambda \rangle = 0 , \tag{22}$$

where Λ is the vector (λ_1, λ_2). These terms are called *resonant.* A system for which all terms are resonant is a *normal form.* Thus, theorem 2 says that every formal system (17) can be put into normal form by a *normalizing transformation.*

Proof. The proof of theorem 2 is given only for the case when $\sigma = 0$. For the proof of the general case see § 1 of Chapter III and Bruno [1964, 1971, 1972a].

With ξ_i given in (21), we differentiate equation (18) with respect to t:

$$\dot{y}_i = \dot{z}_i(1 + h_i) + z_i\left(\frac{\partial h_i}{\partial z_1}\dot{z}_1 + \frac{\partial h_i}{\partial z_2}\dot{z}_2\right) , \qquad i = 1, 2 .$$

Replacing the time derivatives with their expressions in terms of coordinates

from (17), (19), (20), and (21), we obtain

$$\lambda_i y_i + y_i f_i = (\lambda_i z_i + z_i g_i)(1 + h_i) + z_i \frac{\partial h_i}{\partial z_1}(\lambda_1 z_1 + z_1 g_1)$$

$$+ z_i \frac{\partial h_i}{\partial z_2}(\lambda_2 z_2 + z_2 g_2) , \qquad i = 1, 2 .$$

If we now express the y_i in terms of z_1 and z_2 according to formula (18), we get the following system of partial differential equations:

$$\lambda_i z_i + \lambda_i z_i h_i + z_i(1 + h_i) f_i(z_1 + z_1 h_1, z_2 + z_2 h_2)$$

$$= \lambda_i z_i + \lambda_i z_i h_i + z_i g_i + z_i g_i h_i + z_i \frac{\partial h_i}{\partial z_1}(\lambda_1 z_1 + z_1 g_1)$$

$$+ z_i \frac{\partial h_i}{\partial z_2}(\lambda_2 z_2 + z_2 g_2) , \qquad i = 1, 2 .$$

Collecting like terms and moving some of them from one side of the equation to the other gives us

$$\begin{aligned} z_i g_i(Z) + z_i \frac{\partial h_i}{\partial z_1}\lambda_1 z_1 + z_i \frac{\partial h_i}{\partial z_2}\lambda_2 z_2 \\ = -z_i g_i h_i - z_i \frac{\partial h_i}{\partial z_1} z_1 g_1 - z_i \frac{\partial h_i}{\partial z_2} z_2 g_2 \\ + z_i(1 + h_i) f_i(z_1 + z_1 h_1, z_2 + z_2 h_2) , \qquad i = 1, 2 . \end{aligned} \tag{23}$$

By virtue of (18) and (21) we have

$$y_i f_{iS} Y^S = z_i(1 + h_i) f_{iS} Z^S (1 + h_1)^{s_1} (1 + h_2)^{s_2}$$

$$= z_i Z^S f_{iS} \sum h_{1P_1} \dots h_{1P_k} h_{2R_1} \dots h_{2R_l} Z^T ,$$

where

$$T = P_1 + \dots + P_k + R_1 + \dots + R_l .$$

Therefore, in the last term of the right-hand side of (23), the coefficient of $z_i Z^Q$ is a sum of terms of the form

$$f_{iS} h_{1P_1} \dots h_{1P_k} h_{2R_1} \dots h_{2R_l} , \tag{24}$$

where

$$S + P_1 + \dots + P_k + R_1 + \dots + R_l = Q . \tag{25}$$

Equations (23) will be satisfied only if the coefficients of like powers of Z are equal on both sides of the equations. That is, the i^{th} equation of (23) for the

coefficients of $z_i Z^Q$ gives the equations

$$g_{iQ} + h_{iQ}\langle Q, \Lambda\rangle = -\sum_{P+R=Q} h_{iP} g_{iR} - \sum_{P+R=Q} h_{iP}(p_1 g_{1R} + p_2 g_{2R}) \qquad (26)$$

$$+ \text{ sum of terms of form (24)} \; .$$

Note here that h_{iP}, g_{jR}, and f_{is} are non-zero only when $\|P\| > 1$, $\|R\| > 1$, $\|S\| > 1$. But in (25) and (26), the vector Q is a sum of several vectors P, R and S; hence, each of these must satisfy

$$\|P\| < \|Q\| \; , \qquad \|R\| < \|Q\| \; .$$

Let us denote by c_{iQ} the right-hand side of equation (26); c_{iQ} depends only on those h_{jP} and g_{kR} whose vector indices have norms less than that of Q. We thus write equation (26) as

$$g_{iQ} + h_{iQ}\langle Q, \Lambda\rangle = c_{iQ} \; , \qquad Q \in \mathbf{N}_i \; , \qquad i = 1, 2 \; . \qquad (27)$$

We can now find the g_{iQ} and h_{iQ} one-by-one, in the order of the increase of $\|Q\|$. Suppose we already know all the h_{iP} and g_{jR} for $\|P\| < \|Q\|$ and $\|R\| < \|Q\|$; then the c_{iQ} are uniquely determined. We can solve equation (27) thus:

$$\text{if } \langle Q, \Lambda\rangle \neq 0, \text{ then } g_{iQ} = 0 \text{ and } h_{iQ} = c_{iQ}/\langle Q, \Lambda\rangle \; ;$$

$$\text{if } \langle Q, \Lambda\rangle = 0, \text{ then } g_{iQ} = c_{iQ} \text{ and } h_{iQ} \text{ is arbitrary} \; .$$

That is, the coefficients g_{iQ} can be non-zero only for resonant vector indices Q, just as the theorem suggests. The assertion of the theorem is proved by mathematical induction on $\|Q\|$. The induction begins with the linear terms, for which the assertion is trivial. The proof is complete. □

1.6. Classification and Integration of Normal Forms

We will find all normal forms (19). See also Bruno [1971, 1972a, example 1 of the introduction]. In section 1.4 of this chapter we introduced the sets

$$\mathbf{N}_1 = \{Q\text{: either the integers } q_1, q_2 \geqslant 0, \text{ or } q_1 = -1 \text{ and the integer } q_2 \geqslant 1\} \; ,$$

$$\mathbf{N}_2 = \{Q\text{: either the integers } q_1, q_2 \geqslant 0, \text{ or the integer } q_1 \geqslant 1 \text{ and } q_2 = -1\} \; .$$

We now let $\mathbf{N} = \mathbf{N}_1 \cup \mathbf{N}_2$. That is, the set $\mathbf{N}$ consists of those points of the integral lattice in the q_1, q_2 plane which lie either in the first quadrant, in the second quadrant along the half line $q_1 = -1$, $q_2 \geqslant 1$, or in the fourth quadrant along the half-line $q_1 \geqslant 1$, $q_2 = -1$ (figures 47 and 48). Here, $\Lambda = (\lambda_1, \lambda_2)$; let $\lambda_2 \neq 0$, and consider equation (22). If we write $\lambda = \lambda_1/\lambda_2$, this equation is equivalent to

$$\lambda q_1 + q_2 = 0 \; . \qquad (28)$$

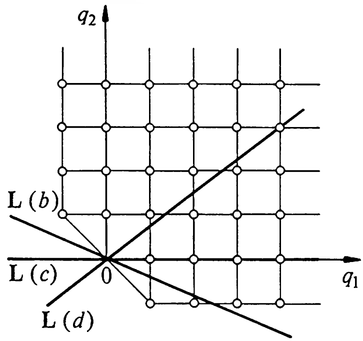

Fig. 48

We consider all possible cases:

a) λ is a complex number (i.e., not real). Then for real q_1 and q_2, equation (28) has only the trivial solution $q_1 = q_2 = 0$, and the normal form is

$$\dot{z}_1 = \lambda_1 z_1 , \qquad \dot{z}_2 = \lambda_2 z_2 . \tag{29}$$

The integral of this system is

$$z_1^{1/\lambda_1} = c z_2^{1/\lambda_2} \quad \text{or} \quad z_2 = c_0 z_1^{1/\lambda} , \tag{29'}$$

and solutions are given by $z_i = c_i \exp \lambda_i t$, $i = 1, 2$.

When λ is real, equation (28) defines a line $\mathbf{L}$ in the q_1, q_2 plane which is orthogonal to the vector $(\lambda, 1)$ and thus also to Λ. The normal form is determined by those points of $\mathbf{N}$ through which this line passes.

b) If $\lambda > 0$, then $\mathbf{L}$ passes through the second and fourth quadrants (in figure 48, $\mathbf{L} = \mathbf{L}(b)$). In the fourth quadrant, the only points of $\mathbf{N}$ are those of the form $q_1 = m > 0, q_2 = -1$. Therefore $\mathbf{L}$ can pass through one of these points only when $\lambda m - 1 = 0$, i.e., $\lambda = m^{-1}$. If $\lambda = m^{-1} < 1$, equation (28) has a unique nontrivial solution $Q = (m, -1) \in \mathbf{N}$ which does not lie in $\mathbf{N}_1$. Hence the corresponding term will appear only in $g_2 = z_2^{-1}\psi_2$. Thus, the normal form is

$$\begin{aligned} \dot{z}_1 &= \lambda_1 z_1 , \qquad \dot{z}_2 = z_2(\lambda_2 + g_{2(m,-1)} z_1^m z_2^{-1}) \quad \text{or} \\ \dot{z}_1 &= \lambda_1 z_1 , \qquad \dot{z}_2 = \lambda_2 z_2 + g_{2(m,-1)} z_1^m . \end{aligned} \tag{30}$$

This system integrates to

$$z_2 = b z_1^m (\ln z_1 + c) , \qquad b = g_{2(m,-1)}/\lambda_1 . \tag{30'}$$

If $\lambda = m > 1$, the result is the same but with the variables exchanged.

If $\lambda = 1$ (i.e., $\lambda_1 = \lambda_2$), the non-trivial solutions of equation (28) in $\mathbf{N}$ are $Q = (-1, 1) \in \mathbf{N}_1$ and $Q = (1, -1) \in \mathbf{N}_2$. These vector exponents correspond to

linear terms on the right-hand side of system (19); their matrix must be a Jordan matrix, so the normal form is a triangular system

$$\dot{z}_1 = \lambda_1 z_1 \ , \qquad \dot{z}_2 = \lambda_2 z_2 + \sigma z_1 \ . \tag{31}$$

If $\sigma = 0$, this is just system (29) with $\lambda_1 = \lambda_2$; if $\sigma \neq 0$, the integral of the system is

$$z_2 = \lambda_1^{-1} z_1 (\ln z_1 + c) \ .$$

If $\lambda \neq m$ and $\lambda \neq m^{-1}$, where m is a natural number, then equation (28) has no non-trivial solutions $Q \in \mathbf{N}$, and the normal form is just system (29).

c) If $\lambda = 0$ (i.e., $\lambda_1 = 0$, $\lambda_2 \neq 0$), the line $\mathbf{L}$ is the q_1 axis and passes through all points $Q = (k, 0) \in \mathbf{N}$, where integer $k \geqslant 0$ (figure 48, $\mathbf{L} = \mathbf{L}(c)$). Consequently, the normal form is

$$\begin{aligned} \dot{z}_1 &= z_1 \sum_{k=1}^{\infty} g_{1(k,0)} z_1^k \equiv z_1 g_1(z_1) \ , \\ \dot{z}_2 &= z_2 \left(\lambda_2 + \sum_{k=1}^{\infty} g_{2(k,0)} z_1^k \right) \equiv z_2 g_2(z_1) \ . \end{aligned} \tag{32}$$

This system is conveniently written as

$$(\ln \dot{z}_1) = g_1(z_1) \ , \qquad (\ln \dot{z}_2) = g_2(z_1) \ .$$

If $g_1 \equiv 0$, then $z_1 = \text{const}$ is the integral of this system. But if $g_1 \not\equiv 0$, then the integral is

$$\ln z_2 = \int \frac{g_2(z_1)\, dz_1}{z_1 g_1(z_1)} + c \ .$$

d) If $\lambda < 0$, the line $\mathbf{L}$ passes through the first and third quadrants (figure 48, $\mathbf{L} = \mathbf{L}(d)$). There are no points of the set $\mathbf{N}$ in the third quadrant, of course, but every point of the integral lattice in the first quadrant is in $\mathbf{N}$. If λ is an irrational number, equation (28) has no nontrivial solutions and the normal form is just system (29). If $\lambda = -1$ (i.e., $\lambda_2 = -\lambda_1$), then $\mathbf{L} \cap \mathbf{N}$ contains every point $Q = (k, k)$, where integer $k \geqslant 0$. The normal form is

$$\begin{aligned} \dot{z}_1 &= z_1 \left(\lambda_1 + \sum_{k=1}^{\infty} g_{1(k,k)} z_1^k z_2^k \right) \equiv z_1 g_1(z_1 z_2) \ , \\ \dot{z}_2 &= z_2 \left(\lambda_2 + \sum_{k=1}^{\infty} g_{2(k,k)} z_1^k z_2^k \right) \equiv z_2 g_2(z_1 z_2) \ . \end{aligned} \tag{33}$$

Under the transformation $w_1 = z_1 z_2$, $w_2 = z_2$, this becomes

$$\dot{w}_1 = w_1 \sum_{k=1}^{\infty} (g_{1(k,k)} + g_{2(k,k)}) w_1^k \ ,$$

$$\dot{w}_2 = w_2\left(\lambda_2 + \sum_{k=1}^{\infty} g_{2(k,k)} w_1^k\right),$$

which is similar to system (32).

When $\lambda = -r/s$, where r and s are relatively prime positive integers, the situation is similar to that above. The line **L** intersects **N** at the points $Q = (ks, kr)$ (again, $k \geqslant 0$ is an integer). The normal form is

$$\begin{aligned} \dot{z}_1 &= z_1\left(\lambda_1 + \sum_{k=1}^{\infty} g_{1(ks,kr)} z_1^{ks} z_2^{kr}\right) \equiv z_1 g_1 , \\ \dot{z}_2 &= z_2\left(\lambda_2 + \sum_{k=1}^{\infty} g_{2(ks,kr)} z_1^{ks} z_2^{kr}\right) \equiv z_2 g_2 . \end{aligned} \tag{34}$$

Since r and s are relatively prime, there exist integers u and v such that $ru - sv = 1$. The power transformation

$$w_1 = z_1^s z_2^r , \qquad w_2 = z_1^u z_2^v \tag{34'}$$

transforms system (34) into the form

$$\begin{aligned} \dot{w}_1 &= w_1 \sum_{k=1}^{\infty} (s g_{1(ks,kr)} + r g_{2(ks,kr)}) w_1^k , \\ \dot{w}_2 &= w_2\left[u\lambda_1 + v\lambda_2 + \sum_{k=1}^{\infty} (u g_{1(ks,kr)} + v g_{2(ks,kr)}) w_1^k\right] , \end{aligned}$$

which is again similar to system (32). Thus, depending on λ_1 and λ_2, the normal forms are systems (29), (30), (31), (32), and (34); the last reduces to (32). In all cases, the normal forms are integrable.

1.7. Power Transformations of Differential Equations

If we write

$$\ln Z = \begin{pmatrix} \ln z_1 \\ \ln z_2 \end{pmatrix} , \qquad \ln W = \begin{pmatrix} \ln w_1 \\ \ln w_2 \end{pmatrix} ,$$

then the power transformation

$$\begin{aligned} w_1 &= z_1^{\alpha_{11}} z_2^{\alpha_{12}} , \\ w_2 &= z_1^{\alpha_{21}} z_2^{\alpha_{22}} , \end{aligned} \qquad \alpha = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} ,$$

can be written conveniently in the matrix form:

$$\ln W = \alpha \ln Z . \tag{35}$$

Likewise, the inverse transformation is

$$\ln Z = \alpha^{-1} \ln W .$$

whence we obtain

$$Z^Q = \exp\langle Q, \ln Z\rangle = \exp\langle Q, \alpha^{-1} \ln W\rangle$$
$$= \exp\langle \alpha^{-1*} Q, \ln W\rangle = W^{\alpha^{-1*}Q} = W^{Q'} ,$$

where α^* is the transpose of α. Thus, the new vector exponent is $Q' = \alpha^{-1*}Q$.

Taking the column vector $G_Q = \begin{pmatrix} g_{1Q} \\ g_{2Q} \end{pmatrix}$, we can write system (19) in vector form:

$$(\ln^{\cdot} Z) = \sum_{Q \in \mathbf{D}} G_Q Z^Q . \tag{36}$$

According to (35),

$$(\ln^{\cdot} W) = \alpha(\ln^{\cdot} Z) = \sum_{Q \in \mathbf{D}} \alpha G_Q Z^Q = \sum_{Q \in \mathbf{D}} \alpha G_Q W^{Q'} ,$$

that is, system (36) is transformed under (35) into

$$(\ln^{\cdot} W) = \sum_{Q' \in \mathbf{D}'} G'_{Q'} W^{Q'} , \tag{37}$$

where $Q' = \alpha^{*-1}Q$ and $G'_{Q'} = \alpha G_Q$. Specifically, the set $\mathbf{D}$ of points Q in the q_1, q_2 plane corresponding to system (36) undergoes a linear transformation with matrix α^{*-1} when the power transformation (35) is applied (just as in the case of a function in section 2.6 of Chapter 1). Therefore, if the points of $\mathbf{D}$ lie entirely on some line $\mathbf{L}$ in the q_1, q_2 plane, then the power transformation can be chosen so that the image of this line is one of the coordinate axes (the q'_1 axis, say). Then all points of the set $\mathbf{D}'$ will have $q'_2 = 0$, and the right-hand side of equation (37) will have no w_2 dependence. This is, in fact, precisely how we integrated normal form (34). This method can also be used to obtain integrals of normal form (30).

1.8. The Convergence of the Normalizing Transformation

We now present without proofs (which are very complicated) results on the convergence of normalizing transformations in cases a)–d) of section 1.6. For proofs, see Bruno [1971, 1972a]. We assume here that the original system is analytic.

In cases a) and b) the normalizing transformation always converges since

$$|\langle Q, \Lambda\rangle| > \varepsilon\|Q\| , \qquad \text{if} \quad Q \in \mathbf{N} , \qquad \langle Q, \Lambda\rangle \neq 0 . \tag{38}$$

In case c) the line $\mathbf{L}$ lies on the q_1 axis. There is an infinite number of points $Q \in \mathbf{N}$ on that line, and the normal form (32) thus has an infinite number of terms. Property (38) does not hold here. The normalizing transformation converges if,

in normal form (32),

$$g_1(z_1) \equiv 0 \ , \tag{39}$$

that is, if all the coefficients $g_{1(k,0)} = 0$. If $g_1 \not\equiv 0$ in the normal form (32), then there exists an analytic system which transforms to this normal form under a divergent transformation (this follows from Theorem III of Bruno, 1971, 1972a). All invariants of the equation (2) with respect to analytic transformations of variables may be found in Martinet and Ramis [1982].

Example 4 (Continuation of example 3). The system

$$\dot{x}_1 = x_1^2 \ , \qquad \dot{x}_2 = x_2 - x_1 \tag{40}$$

is equivalent to the equation

$$x_1^2 \frac{dx_2}{dx_1} = x_2 - x_1 \ . \tag{41}$$

The transformation

$$x_1 = y_1 \ , \qquad x_2 = y_2 + \sum_{k=1}^{\infty} (k-1)! y_1^k = y_2 + \xi(y_1) \tag{42}$$

transforms system (40) into

$$\dot{y}_1 = y_1^2 \ , \qquad \dot{y}_2 = y_2 \ . \tag{43}$$

This corresponds to the fact, that using the particular solution $x_2 = \xi(y_1)$ of the inhomogeneous equation (41), we have put it into homogeneous form. But in normal form (43), we have $g_1 = y_1 \not\equiv 0$; that is, requirement (39) is not satisfied, and the normalizing transformation (42) diverges.

In case d), for rational $\lambda = -r/s$, the normal form takes form (34). If there exists a series

$$a(z_1, z_2) = \sum_{k=0}^{\infty} a_k z_1^{ks} z_2^{kr} \ , \tag{44}$$

such that

$$g_i = \lambda_i a \ , \qquad i = 1, 2 \ , \tag{45}$$

in normal form (34), then the normalizing transformation converges. If condition (45) is not met, then there exists an analytic system which is transformed to normal form (34) under a divergent normalizing transformation. Condition (45) means that all the vectors G_Q in system (34) are proportional to the vector Λ, and $g_1 s + g_2 r = 0$. Under power transformation (34′), system (34) becomes

$$\begin{aligned} (\ln w_1)^{\cdot} &= s g_1 + r g_2 \equiv g_1' \ , \\ (\ln w_2)^{\cdot} &= u g_1 + v g_2 \equiv g_2' \ , \end{aligned} \tag{46}$$

for which condition (45) implies that $g_1' \equiv 0$. System (46) is analogous to system

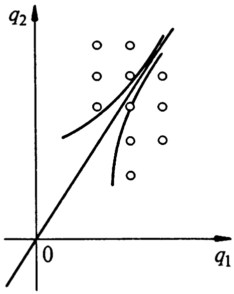

Fig. 49

(32), and has the same condition for convergence; it is invariant under power transformations. Geometrically, condition (45) just means that the vectors G_Q are orthogonal to the line **L**. For analytical invariants see Martinet and Ramis [1983].

In case d) with irrational λ, the normal form (29) is simple, but the conditions for the convergence of the normalizing transformation are not. In fact, the question of convergence cannot be completely answered in this case. Siegel [1952] proved convergence when all the integral vectors Q satisfy

$$|\langle Q, \Lambda \rangle| > \varepsilon(|q_1| + |q_2|)^{-\nu} , \tag{47}$$

where ν is some positive number. Geometrically, this means that the integral points Q do not approach too close to the line **L** (figure 49). Condition (47) can be slightly relaxed in order to preserve convergence. On the other hand, if those integral points do approach too close to the line **L**, then there will be divergence. The exact boundary (with respect to Λ) between convergence and divergence has not yet been established; but as of this writing, sufficient conditions for convergence and sufficient conditions for divergence are known; they are very close to each other (see Bruno, 1971, 1972a).

1.9. Real Systems

Until now, we have been concerned with complex systems (1); that is, systems in which the coefficients in the series φ_i, as well as the coordinates x_i, are complex-valued. Further, we have used complex changes of coordinates to simplify these systems.

But now we wish to consider specifically real systems (1), i.e., those in which the coefficients of the series φ_i, and the coordinates x_i, are real-valued. For such systems, it is natural to seek simplifications with real changes of coordinates.

Thus, we consider the real system

$$\dot{X} = AX + \Theta(X) , \tag{48}$$

where the series Θ contains no constant or linear terms, and the coefficients of the series Θ and the elements of the matrix A are all real numbers. The characteristic equation for A, $\det(A - \lambda E) = 0$, has real coefficients, so the eigenvalues λ_1 and λ_2 of A (the roots of the characteristic equation) are either both real or both complex. If they are real, then the linear part of system (48) can be put in Jordan form by a real linear transformation. As we saw in the proof of theorem 2, the transformation to normal form will also have real coefficients if the resonant coefficients h_{iQ} are taken as real. Consequently, the normalizing transformation and the normal form will both be real.

If the eigenvalues are complex, they must be complex conjugates

$$\lambda_1 = \bar{\lambda}_2 = \mu + i\nu\ , \qquad \nu \neq 0\ . \tag{49}$$

In this case we call a system

$$\dot{v}_j = \eta_j(v_1, v_2)\ , \qquad j = 1, 2 \tag{50}$$

a *real normal form* if the system is an ordinary normal form in the complex-conjugate coordinates

$$z_1 = v_1 + iv_2\ , \qquad z_2 = v_1 - iv_2 \tag{51}$$

Theorem 3. *A real system* (48) *with complex-conjugate eigenvalues* (49) *can be transformed into a real normal form* (50) *with the help of a real, formal coordinate change* (3).

Proof. By virtue of (49), $\lambda_1 \neq \lambda_2$, so the Jordan form J of the matrix A is diagonal. Let B_1 be the eigenvector of A corresponding to the eigenvalue λ_1; then its complex conjugate vector, $\bar{B}_1$, will be the eigenvector corresponding to the other eigenvalue, $\lambda_2 = \bar{\lambda}_1$. We write the matrix B as $(B_1, \bar{B}_1)$. We saw in section 1.3 of this chapter that under the linear transformation

$$X = BY \tag{52}$$

the matrix A becomes the Jordan matrix $J = B^{-1}AB$. Real values of variables X then correspond to complex-conjugate values of variables Y:

$$y_1 = \bar{y}_2\ . \tag{53}$$

For a series $f(Y) = \sum f_Q Y^Q$, we will let $\bar{f}(Y) = \sum \bar{f}_Q Y^Q$.

Under transformation (52), system (48) becomes

$$\dot{y} = \lambda_i y_i + y_i f_i(y_1, y_2)\ , \qquad i = 1, 2\ . \tag{54}$$

Since the variables y_i are complex-conjugates, the second equation of (54) is conjugate to the first, and

$$\lambda_1 y_1 + y_1 f_1(y_1, y_2) = \bar{\lambda}_2 \bar{y}_2 + \bar{y}_2 \bar{f}_2(\bar{y}_1, \bar{y}_2)\ .$$

Taking equation (53) into account, we obtain the correspondence $f_1(y_1, y_2) = \bar{f}_2(y_2, y_1)$, which yields the following relation between the coefficients in the series f_1 and f_2:

$$f_{1(q_1,q_2)} = \bar{f}_{2(q_2,q_1)} \ . \tag{55}$$

Similarly, if the formal change of coordinates

$$y_i = z_i + z_i \sum h_{iQ} Z^Q \ , \qquad i = 1, 2 \tag{56}$$

satisfies

$$h_{1(q_1,q_2)} = \bar{h}_{2(q_2,q_1)} \ , \tag{57}$$

then the complex-conjugate coordinates y_1 and y_2 correspond to complex-conjugate variables z_1 and z_2 (i.e., $z_1 = \bar{z}_2$). We will show that there exists a normalizing transformation (56) with property (57), which transforms system (54) into the normal form

$$\dot{z}_i = \lambda_i z_i + z_i \sum g_{iQ} Z^Q \ , \qquad i = 1, 2 \ , \tag{57'}$$

where

$$g_{1(q_1,q_2)} = \bar{g}_{2(q_2,q_1)} \ . \tag{58}$$

We will prove this by mathematical induction on the norm $\|Q\|$. Let $Q = (q_1, q_2)$, $Q' = (q_2, q_1)$, and let properties (57) and (58) be satisfied for all coefficients h_{jP} and g_{kR} for which $\|P\| < \|Q\|$ and $\|R\| < \|Q\|$. We will show that the equations

$$h_{1Q} = \bar{h}_{2Q'} \ , \qquad g_{1Q} = \bar{g}_{2Q'} \tag{59}$$

are satisfied. Recall that the coefficients in the series f, g, and h are related by equations (26). The right-hand sides of these equations, which we denoted by c_{iQ}, will thus satisfy

$$c_{1Q} = \bar{c}_{2Q'} \ . \tag{60}$$

This follows from property (55) and from the inductive hypotheses of properties (57) and (58). Equations (27) yield

$$g_{1Q} + h_{1Q}\langle Q, \Lambda\rangle = c_{1Q} \ , \qquad g_{2Q'} + h_{2Q'}\langle Q', \Lambda\rangle = c_{2Q'} \ .$$

In agreement with (49)

$$\langle Q, \Lambda\rangle = \mu(q_1 + q_2) + i\nu(q_1 - q_2) \ ,$$

$$\langle Q', \Lambda\rangle = \mu(q_1 + q_2) + i\nu(q_2 - q_1) \ .$$

That is, $\langle Q, \Lambda\rangle = \overline{\langle Q', \Lambda\rangle}$. Therefore, if $\langle Q, \Lambda\rangle \neq 0$, then

$$g_{1Q} = \bar{g}_{2Q'} = 0 \ , \qquad h_{1Q} = \frac{c_{1Q}}{\langle Q, \Lambda\rangle} \ , \qquad h_{2Q'} = \frac{c_{2Q'}}{\langle Q', \Lambda\rangle} \ ,$$

and, thanks to (60), property (59) is fulfilled. If $\langle Q, \Lambda\rangle = 0$, then $g_{1Q} = \bar{g}_{2Q'} = c_{1Q}$,

the coefficient h_{1Q} can be arbitrarily chosen, and we take $h_{2Q'} = \bar{h}_{1Q}$. Thus, property (59) is satisfied for all Q, and system (57′) is a normal form. Transformation (51) from Z to V coordinates then transforms the complex normal form (57′) into a real normal form (50). Finally, note that the successive application of transformations (52), (56) and (51) results in a real transformation from X to V coordinates. The proof is complete. □

For the general case, the proof of the existence of a real normalizing transformation is given in section 1.9 of Chapter III.

For a discussion of the divergence of such transformations, see Bruno [1982] and Martinet and Ramis [1986].

1.10. On a Smooth Normalization

For real coordinate changes, it is possible to use not only analytic transformations, but also smooth (i.e., continuously differentiable), or just continuous, transformations. Since there frequently exists no analytic normalizing transformation, the following problems arise for real systems:

1) In what cases does there exist an infinitely differentiable transformation to a real normal form?

2) In what cases can the system be linearized by means of a continuously differentiable change of coordinates?

3) More generally: what is the simplest system to which a given system can be transformed with the help of change of coordinates of a finite degree of smoothness.

At present, we have the following answers to the above questions:

1) This question was posed by Birkhoff [1929]. Sternberg [1958, 1959] and Chen [1963, 1965] both arrived at the result that, provided that

$$\operatorname{Re}\lambda_1 \neq 0\ , \qquad \operatorname{Re}\lambda_2 \neq 0 \tag{60'}$$

there always exists an infinitely smooth normalizing transformation. Bruno [1973c] presents an hypothesis which, for two-dimensional systems, can be formulated thus:

Hypothesis. *There always exists an infinitely smooth normalizing transformation for a two-dimensional, real, analytic system.*

Tokarev [1977], refining a result of Bibikov [1971], proved the validity of this hypothesis in the case of pure imaginary $\lambda_1 = \bar{\lambda}_2 = -\lambda_2$, when condition (44) is not satisfied. Tokarev [1979, 1984] and Belitskiĭ [1986] proved the general case.

2) Under condition (60′), every two-dimensional system is equivalent to its linear part in the class of continuously differentiable changes of variables (see Hartman, 1964, Ch. 9, 148). Samovol [1972] formulated the conditions for a

linearization for an infinitely smooth system in the n-dimensional case by means of a transformation of a finite order of smoothness. Belitskiĭ [1973, 1975b, 1978, 1979b] studies conditions for a linearization of a finitely smooth system by means of finitely smooth transformations. If condition (60′) is not satisfied, it is easy to construct examples in which the integral curves of the normal form and of the linear system are not even topologically equivalent.

3) There are no complete answers to this question, though there are some isolated results (see Samovol, 1979, 1982).

1.11. Integrating Real Normal Forms

In the same order as in section 1.6 of this chapter, we will analyze the behavior of the integral curves of normal forms of real systems (see also sec. 1.8).

a) λ is a complex number. Then $\operatorname{Re}\lambda_1 = \mu \neq 0$. The normal form is the system

$$\dot{z}_1 = \lambda_1 z_1 \ , \qquad \dot{z}_2 = \bar{\lambda}_1 z_2 \ .$$

The real normal form, in the variables of (51), is

$$\dot{v}_1 = \mu v_1 - \nu v_2 \ , \qquad \dot{v}_2 = \nu v_1 + \mu v_2 \ .$$

Its solutions are

$$v_1 = c e^{\mu t} \cos \nu t \ , \qquad v_2 = c e^{\mu t} \sin \nu t \ . \tag{61}$$

These integral curves are spirals, approaching the origin in the v_1, v_2 plane (figure 50). This disposition of the integral curves is called a *focus*. So long as the normalizing transformation in this case converges, then the integral curves of the given system will be slightly distorted spirals.

b) λ is a real, positive number. In this case, λ_1 and λ_2 are also real. The normal form may be of type (29), or (if $\lambda = m^{-1}$) type (30). For system (29) with $\lambda_1 \neq \lambda_2$, the integral curves $z_2 = c z_1^{\lambda_2/\lambda_1}$ converge to the singular point, tangent to the z_i axis if $|\lambda_i|$ is the smaller of $|\lambda_1|$ and $|\lambda_2|$ (figure 51). Moreover, both axes are integral curves. If $\lambda_1 = \lambda_2$, the integral curves are lines $z_2 = c z_1$ (figure 52).

If $\lambda = m^{-1}$, and in the normal form (30) $g_{2(m_1-1)} \neq 0$, then for $m > 1$ the disposition of the integral curves (30′) is similar to figure 51, except that the z_1

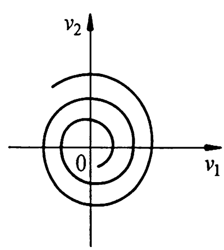

Fig. 50

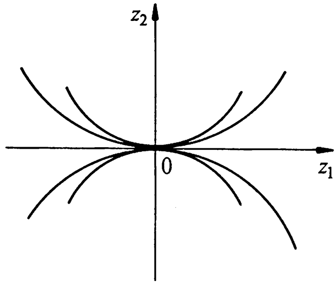

Fig. 51

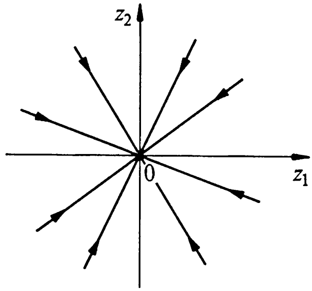

Fig. 52

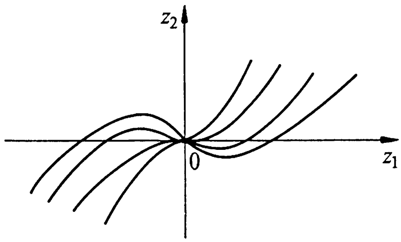

Fig. 53

axis is not an integral curve (figure 53). Note that in this case the smooth transformation

$$w_2 = z_2 - bz_1^m \ln|z_1| \tag{62}$$

transforms system (30) into (29); that is, these cases are equivalent in the class of smooth changes of variables.

If $\lambda = 1$ and $\sigma \neq 0$ in system (31), then the logarithm will be dominant in the integral curves $z_2 = \sigma\lambda_1^{-1} z_1(\ln|z_1| + c)$ (figure 54). Transformation (62) is no

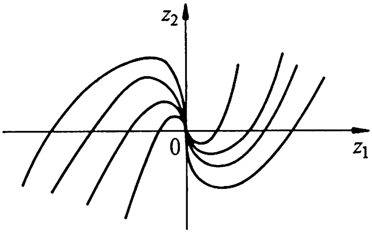

Fig. 54

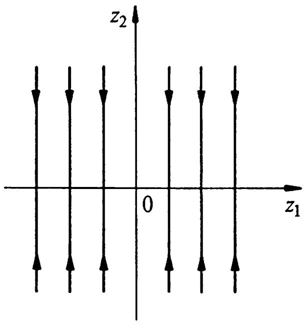

Fig. 55

longer smooth, but it remains continuous and takes system (31) into form (29). Since the normalizing transformation always converges when $\lambda > 0$, all real systems in this case are equivalent to system (29) from a topological point of view. This case is called a *node*.

c) $\lambda = 0$. Here $\lambda_1 = 0$, and λ_2 is a real and non-zero number. The normal form has the form of system (32). If condition (39) is satisfied, the normalizing transformation converges, and the solutions of the normal form are $z_1 = c = \text{const}$, $z_2 = \exp g_2(c)t$. The integral curves are vertical half-lines (figure 55), and the z_1 axis consists of fixed points.

If $g_1 \not\equiv 0$, then $g_1 = g_{1(l,0)} z_1^l + \cdots$; the integral of the system (32) is

$$\ln|z_2| + c = \int \frac{g_2(z_1)\,dz_1}{z_1 g_1(z_1)} = \int \frac{\lambda_2}{g_{1(l,0)} z_1^{l+1}} (1 + \cdots)\,dz_1$$

$$= -\frac{\lambda_2}{l g_{1(l,0)} z_1^l} (1 + \cdots) ,$$

since $g_2 = \lambda_2 + \cdots$. Here, the z_1 and z_2 axes are always integral curves. The behavior of the integral curves depends on the parity of l and the sign of the

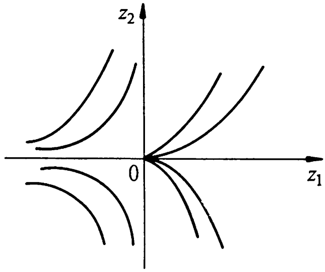

Fig. 56

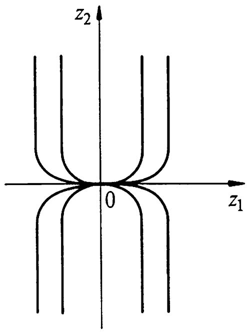

Fig. 57

ratio $\lambda_2/g_{1(l,0)}$. If l is odd, the integral curves behave exactly as in figure 56, up to a change of sign in the z_1 coordinate (a *saddle-node*). If l is even, then the behavior is as in figure 57 for $\lambda_2/g_{1(l,0)} > 0$, or figure 58 for $\lambda_2/g_{1(l,0)} < 0$. In figures 56 and 57, the tangency of the integral curves to the z_1 axis is of infinite order.

Although the normalizing transformation diverges in cases when $g_1 \not\equiv 0$, there always exists a real, infinitely differentiable normalizing transformation. Belitskiĭ [1986] has proven this (see the hypothesis in § 1.10 above).

d) $\lambda < 0$. There are two principal sub-cases.

First subcase: the numbers λ_1 and λ_2 are real. Then the integral curves of the normal form are hyperbolas (figure 59—a *saddle* point). As Sternberg [1958, 1959] showed, there always exists a real, infinitely differentiable transformation into a normal form in this case (see also Birkhoff, 1929 and Chen, 1963). If λ is irrational, the normal form is a linear system (29). For rational λ, the normal form (34) contains, generally speaking, infinite series. But with the help of a formal change of coordinates, it is possible to truncate the right-hand sides of system

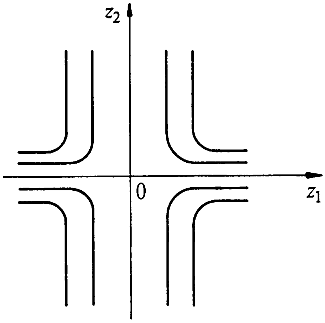

Fig. 58

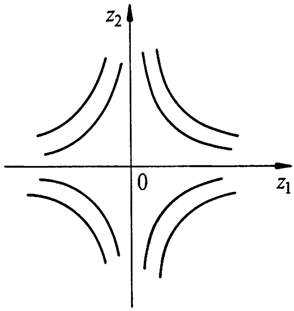

Fig. 59

(34) to a finite number of terms (see section 2.3 of Chapter III). In the class of continuously differentiable changes of coordinates, the normal form is always equivalent to a linear system (see the preceding section).

The second subcase: the numbers λ_1 and λ_2 are complex, hence pure imaginary: $\lambda_1 = -\lambda_2 = i\nu$. Here $\lambda = -1$, and the normal form is system (33). This is integrated by introducing the variable $w_1 = z_1 z_2 = v_1^2 + v_2^2$, since

$$\dot{w}_1 = w_1(g_1(w_1) + g_2(w_1)) .$$

Here the reality condition $g_{1(q_1,q_2)} = \bar{g}_{2(q_2,q_1)}$ implies that $g_{1(k,k)} = \bar{g}_{2(k,k)}$, that is, that all coefficients of the series $g_1(w_1) + g_2(w_1) = 2\sum \operatorname{Re} g_{1(k,k)} w_1^k$ are real. The variable w_1 is also real, and will be the square of the length of the radius vector in the real plane $v_1 = \operatorname{Re} z_1$, $v_2 = \operatorname{Im} z_1$: $w_1 = z_1 \bar{z}_1 = (\operatorname{Re} z_1)^2 + (\operatorname{Im} z_1)^2$.

If $g_1 + g_2 \equiv 0$—that is, if all the coefficients $g_{1(k,k)}$ are pure imaginary—then the system integrates to $w_1 = \text{const.}$, and the integral curves in the v_1, v_2 plane

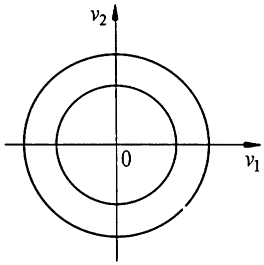

Fig. 60

are the circles $v_1^2 + v_2^2 = \text{const.}$ (figure 60). In this case (called a *center*) the normalizing transformation always converges.

If $g_1 + g_2 \not\equiv 0$, then $g_1 + g_2 = aw_1^l + \cdots$, where the number a is real and non-zero, $l \geqslant 1$, the coefficients $g_{1(k,k)}$ are pure imaginary for $k < l$, and $a = 2\operatorname{Re} g_{1(l,l)} \neq 0$. The integral of this system for real w_1 is

$$\ln z_1 = \int \frac{g_1(w_1)\,dw_1}{w_1(g_1 + g_2)} = \int \frac{\operatorname{Re} g_1\,dw_1}{w_1(g_1 + g_2)} + i\int \frac{\operatorname{Im} g_1\,dw_1}{w_1(g_1 + g_2)},$$

but

$$\frac{\operatorname{Re} g_1}{g_1 + g_2} = \frac{1}{2}, \qquad \frac{\operatorname{Im} g_1}{g_1 + g_2} = \frac{\operatorname{Im} \lambda_1}{aw_1^l}(1 + \cdots).$$

Therefore, the above integral is

$$\ln z_1 = \frac{1}{2}\ln w_1 - \frac{i\operatorname{Im} \lambda_1}{alw_1^l}(1 + \cdots).$$

That is, the integral curves are spirals: as $w_1 \to 0$, they turn clockwise if $\operatorname{Im} \lambda_1/a > 0$, counterclockwise if $\operatorname{Im} \lambda_1/a < 0$.

This case is called a *non-structurally stable focus*; its existence and character are determined by the nonlinear terms of the normal form. In this case, the normalizing transformation diverges, but there exists an infinitely smooth normalizing transformation (see Tokarev, 1977).

Necessary and sufficient conditions for the existence of a center are the countable number of the conditions

$$g_{1(k,k)} + g_{2(k,k)} = 0, \qquad k = 1, 2 \ldots$$

Here, the $g_{j(k,k)}$ are polynomials in the coefficients of the given system (48). If the right-hand sides of the equations in that system are polynomials of degree m, then it is sufficient that a finite number of those conditions be met (only for $k \leqslant n(m)$). Thus, $n(2) = 3$ (see Sibirskiĭ, 1976, Siegel, 1956, § 25).

Remark. Normalizing transformations may be used to integrate differential equations approximately. To do this for system (1), only terms up to some finite degree are reduced to a normal form; all remaining terms are discarded. The solution of such a "truncated" system will be an approximate solution of the entire system if the "truncated" normal form contains leading terms (for example, the term $g_{1(l,0)}z_1^l$ in the case c), or an analogous term in the case of a non-structurally stable focus).

Exercise 2. For each of the systems given below, find the normal form, integrate it, and sketch its integral curves.

1) $\dot{x} = ax - 4y + (x + y)^4$, $\dot{y} = x - (x - y)^4$, $a = 6, 5, 4, 3$.

2) $\dot{x} = 2x + y + 7x^2 + 3y^2$, $\dot{y} = -x + 6x^2 + y^2$.

For the systems below, find the normal form only to those terms necessary to determine the character of the integral curves; find the behavior of those curves.

3) $\dot{x} = 3x + 4y + 17x^2 + xy$, $\dot{y} = x - 23y^2$.

4) $\dot{x} = 2x + y + 2x^2 + y^2$, $\dot{y} = 6x + 3y + 3x^2 + xy$.

5) $\dot{x} = y + 6x^3 + xy^2$, $\dot{y} = -4x - 2x^2y + y^3$.

6) For what values of μ and ν does solution (61) approach the origin as $t \to +\infty$, and in what direction does the spiral turn in the v_1, v_2 plane?

§ 2. Generalized Normal Forms

2.1. The Second Theorem on the Normal Form

Up to this point, we have considered systems

$$\dot{x}_i = \varphi_i(x_1, x_2) \ , \qquad i = 1, 2 \ ,$$

in which the φ_i have been series in non-negative powers of the variables. This reduced to the form

$$\dot{x}_i = \lambda_i x_i + x_i \sum f_{iQ} X^Q \equiv \lambda_i x_i + x_i f_i \ , \qquad i = 1, 2 \ , \tag{1}$$

where the vector exponents Q ran through a set $\mathbf{N}$ almost entirely contained in the first quadrant of the $\mathbf{R}_1^2$ plane. For each vector Q, we form a vector coefficient $F_Q = (f_{1Q}, f_{2Q})$ and create a set $\mathbf{D} = \mathbf{D}(F) = \{Q\colon F_Q \neq 0\}$. That is, $\mathbf{D}(F)$ is the set of vector exponents of system (1) (the support of F). For example, if the functions φ_i are divisible by x_i, then the set $\mathbf{D}$ for system (1) lies entirely in the first quadrant.

We now consider a more general situation, in which the set $\mathbf{D}(F)$ lies in some sector $\mathbf{V}$ of the $\mathbf{R}_1^2$ plane, bounded by rays extending from the origin (figure 61; compare with section 2.5 of Chapter I and figure 28). Thus, let R^* and R_* be two linearly independent vectors, and let P^* and P_* be vectors orthogonal to R^* and R_*, respectively, with

$$\langle R^*, P_* \rangle < 0 \ , \qquad \langle R_*, P^* \rangle < 0 \ . \tag{2}$$

We define the sector $\mathbf{V}$ of the $\mathbf{R}_1^2$ plane as the set of points Q such that

$$\langle Q, P_* \rangle \leqslant 0 \ , \qquad \langle Q, P^* \rangle \leqslant 0 \ . \tag{3}$$

The boundary of the sector $\mathbf{V}$ consists of the rays passing through the points R^* and R_*. Because of condition (2), the sector $\mathbf{V}$ lies to one side of the line passing through the origin and R^* (likewise R_*); that is, the angle of the sector $\mathbf{V}$ is less than π, and $\mathbf{V}$ is a convex cone. Therefore every point Q in $\mathbf{V}$ can be expressed in the form

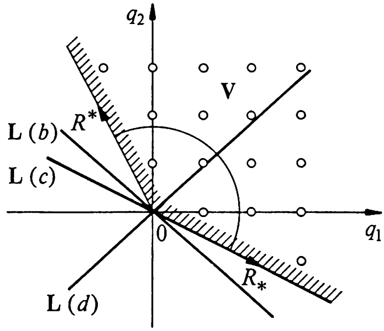

Fig. 61

$$Q = \alpha_1 R^* + \alpha_2 R_* , \qquad \alpha_1 \geqslant 0 , \qquad \alpha_2 \geqslant 0 . \tag{4}$$

We introduce the vector $T = -P^* - P_*$. The equation $\langle Q, \mathrm{T} \rangle = 0$ defines a line through the origin in the q_1, q_2 plane; the sector **V** lies entirely on one side of this line, since for $Q \neq 0$ at least one of the scalar products $-\langle Q, \mathrm{P}^* \rangle$ and $-\langle Q, \mathrm{P}_* \rangle$ must be non-zero by the linear independence of the vectors P^* and P_*. But if $Q \in \mathbf{V}$, condition (3) informs us that these products are non-negative. Hence, for $Q \in \mathbf{V} \backslash 0$ we have

$$\langle Q, T \rangle = -\langle Q, P_* \rangle - \langle Q, P^* \rangle > 0 .$$

We denote by $\mathscr{V}(X)$ the class of all series of the form

$$f = \sum f_Q X^Q , \tag{5}$$

where the integral vector exponents Q lie in the sector **V**. Note that sums and products of series of class $\mathscr{V}(X)$ also belong to this class. If $|X|^{R^*} \to 0$ and $|X|^{R_*} \to 0$ then, by (4), $X^Q \to 0$, for all $Q \in \mathbf{V} \backslash 0$. We therefore call series (5) *convergent* if it converges absolutely for some $\varepsilon > 0$ on the set

$$\mathscr{U}_{\mathbf{V}}(\varepsilon) = \{X: |X|^{R_*} \leqslant \varepsilon, |X|^{R^*} \leqslant \varepsilon, |x_1| \leqslant \varepsilon, |x_2| \leqslant \varepsilon\} .$$

Remark. Suppose a series (5) belongs to the class $\mathscr{V}$, has no constant term, and converges on the set $\mathscr{U}_{\mathbf{V}}(\varepsilon)$. If we calculate $\gamma(\varepsilon) = \max |f(X)|$ for $X \in \mathscr{U}_{\mathbf{V}}(\varepsilon)$, then $\gamma(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Lemma 1. *If in system* (1) *the f_i are convergent series of class $\mathscr{V}(X)$, then every coordinate transformation*

$$x_i = y_i(1 + h_i) \equiv y_i + y_i \sum h_{iQ} Y^Q , \qquad i = 1, 2 , \tag{6}$$

where the h_i are convergent series of class $\mathscr{V}(Y)$, transforms system (1) *into the system*

$$\dot{y}_i = \lambda_i y_i + y_i g_i \equiv \lambda_i y_i + y_i \sum g_{iQ} Y^Q\ , \qquad i = 1, 2\ , \tag{7}$$

where the g_i are also convergent series of class $\mathscr{V}(Y)$.

Proof. 1) If system (1), transformation (6), and system (7), are written in vector form,

$$\dot{X} = \Phi(X)\ , \tag{8}$$

$$X = Y + \Xi(Y)\ , \tag{9}$$

$$\dot{Y} = \Psi(Y)\ , \tag{10}$$

then differentiating equation (9) gives

$$\dot{X} = \dot{Y} + \frac{\partial \Xi}{\partial Y}\dot{Y} = \left(E + \frac{\partial \Xi}{\partial Y}\right)\dot{Y}\ ,$$

Employing expressions (8), (10), and once again (9), we obtain

$$\left(E + \frac{\partial \Xi}{\partial Y}\right)\Psi(Y) = \Phi(Y + \Xi)$$

or

$$\Psi(Y) = \left(E + \frac{\partial \Xi}{\partial Y}\right)^{-1}\Phi(Y + \Xi)\ . \tag{10'}$$

2) Let us say that a vector $W = (w_1, w_2)$ is of class $\mathscr{V}^*$ if $w_i y_i^{-1} \in \mathscr{V}$. For example, Ξ is a (convergent) vector of class $\mathscr{V}^*$.

Our problem is to show that Ψ is a vector of class $\mathscr{V}^*$. First, we shall show that the vector $\Phi(Y + \Xi(Y))$ is of class $\mathscr{V}^*$. Returning to the notation of (1) and (6), we see that

$$\begin{aligned}\varphi_i &= x_i f_i(X) = x_i \sum f_{iS} X^S \\ &= y_i(1 + h_i)\sum f_{iS}(1 + h_1)^{s_1}(1 + h_2)^{s_2} Y^S\ , \qquad i = 1, 2\ .\end{aligned}$$

Note that for a convergent series $h_i \in \mathscr{V}$, the series $(1 + h_i)^{s_i}$ will be a convergent series of class $\mathscr{V}$ for any integer s_i. In fact, if $s_i > 0$, then $(1 + h_i)^{s_i}$ is a polynomial in h_i and is, consequently, a convergent series of class $\mathscr{V}$ together with h_i. And if $s_i < 0$, then for $|h_i| < 1$ the expression $(1 + h_i)^{s_i}$ can be expanded in a convergent series of positive integer powers of h_i—once again, a convergent series of class $\mathscr{V}$. The inequalities $|h_i| < 1$ will be satisfied for series h_i which, for sufficiently small ε, converge on the set

$$\mathscr{U}_V(\varepsilon) = \{Y : |Y|^{R_*} \leqslant \varepsilon, |Y|^{R^*} \leqslant \varepsilon, |y_1| \leqslant \varepsilon, |y_2| \leqslant \varepsilon\}\ . \tag{11}$$

Therefore the product $Y^S(1+h_1)^{s_1}(1+h_2)^{s_2}$ is a convergent series of class $\mathscr{V}$, since $S \in \mathbf{V}$. So, too, is the series

$$(1+h_i)\sum f_{is} Y^S(1+h_1)^{s_1}(1+h_2)^{s_2}\ ,$$

that is, the vector $\Phi(Y+\Xi(Y))$ is of class $\mathscr{V}^*$. By the remark preceding the lemma, $|h_i| < 1/2$ for $Y \in \mathscr{U}_{\mathbf{V}}(\varepsilon)$ when $\varepsilon < \varepsilon_0$; hence

$$\begin{aligned} |X|^{R^*} &= |x_1|^{r_1^*}|x_2|^{r_2^*} = |Y|^{R^*}|1+h_1|^{r_1^*}|1+h_2|^{r_2^*} \\ &< |Y|^{R^*}|1-\tfrac{1}{2}|^{r_1^*}|1+\tfrac{1}{2}|^{r_2^*} \leqslant \varepsilon 2^{-r_1^*}(\tfrac{3}{2})^{r_2^*} = c^*\varepsilon\ . \end{aligned}$$

Similarly,

$$|X|^{R_*} < \varepsilon(\tfrac{3}{2})^{r_{1*}}2^{-r_{2*}} = c_*\varepsilon\ .$$

That is, $X \in \mathscr{U}_{\mathbf{V}}(\varepsilon_1)$, where $\varepsilon_1 = \varepsilon\max[c^*, c_*]$. Thus, $X \in \mathscr{U}_{\mathbf{V}}(\varepsilon_1)$ for $Y \in \mathscr{U}_{\mathbf{V}}(\varepsilon)$. So long as the vector $\Phi(X)$ converges for $X \in \mathscr{U}_{\mathbf{V}}(\varepsilon_1)$, the vector $\Phi(Y+\Xi(Y))$ also converges for $Y \in \mathscr{U}_{\mathbf{V}}(\varepsilon)$ for sufficiently small ε. Therefore $\Phi(Y+\Xi(Y))$ is a convergent vector of class $\mathscr{V}^*$.

3) A matrix $A = (a_{ij})$ is said to be of class $\mathscr{V}^{**}$ if $a_{ij}y_i^{-1}y_j \in \mathscr{V}$. Thus, the matrix

$$\frac{\partial \Xi}{\partial Y} = \begin{pmatrix} \dfrac{\partial \xi_1}{\partial y_1} & \dfrac{\partial \xi_1}{\partial y_2} \\[2ex] \dfrac{\partial \xi_2}{\partial y_1} & \dfrac{\partial \xi_2}{\partial y_2} \end{pmatrix}$$

belongs to class $\mathscr{V}^{**}$. Further, we have

$$\left(E + \frac{\partial \Xi}{\partial Y}\right)^{-1} = \frac{1}{\Delta}\begin{pmatrix} 1 + \dfrac{\partial \xi_2}{\partial y_2} & -\dfrac{\partial \xi_1}{\partial y_2} \\[2ex] -\dfrac{\partial \xi_2}{\partial y_1} & 1 + \dfrac{\partial \xi_1}{\partial y_1} \end{pmatrix}, \tag{11'}$$

where

$$\Delta = \det\left(E + \frac{\partial \Xi}{\partial Y}\right) = \left(1 + \frac{\partial \xi_1}{\partial y_1}\right)\left(1 + \frac{\partial \xi_2}{\partial y_2}\right) - \frac{\partial \xi_1}{\partial y_2}\frac{\partial \xi_2}{\partial y_1}\ ,$$

and the matrix to the right of $1/\Delta$ belongs to class $\mathscr{V}^{**}$. Clearly, Δ is a convergent series of class $\mathscr{V}$ with a constant term 1; i.e., $\Delta = 1 + \tilde{\Delta}(Y)$. By the remark preceding the lemma, if $Y \in \mathscr{U}_{\mathbf{V}}(\varepsilon)$ for sufficiently small ε, then $|\tilde{\Delta}| < 1$. Therefore Δ^{-1} can be expanded in a convergent series of positive integral powers of $\tilde{\Delta}$, so that Δ^{-1} is a convergent series of class $\mathscr{V}$, consequently matrix (11′) is a convergent matrix of class $\mathscr{V}^{**}$.

4) Finally, we note that the product $W = AZ$ of a vector Z of class $\mathscr{V}^*$ by a matrix A of class $\mathscr{V}^{**}$ is again a vector of class $\mathscr{V}^*$. In fact, $w_i = \sum a_{ij}z_j = y_i\sum(a_{ij}y_i^{-1}y_j)(z_jy_j^{-1})$, where the expressions in parentheses are class $\mathscr{V}$ series.

Consequently, the product

$$\left(E + \frac{\partial \Xi}{\partial Y}\right)^{-1} \Phi(Y + \Xi(Y))$$

will be a convergent vector of class $\mathscr{V}^*$. But this is just $\Psi(Y)$. The lemma is proved. □

We will now consider formal series of class $\mathscr{V}$, which may even diverge on any set $\mathscr{U}_{\mathrm{V}}(\varepsilon)$. Since the arithmetic operations and differentiation are the same for formal series as for convergent series, we can conclude the following from Lemma 1.

If in system (1) and transformation (6), the f_i and h_i are formal series of class $\mathscr{V}$, then the g_i in system (7) are likewise formal series of class $\mathscr{V}$. We then face the problem of transforming the formal system (1) into the simplest possible system (7) with the help of a formal change of coordinates (6). The answer to this problem is given by

Theorem 1. *Every formal series* (1) *can be transformed by a formal variable change* (6) *into a system* (7) *in which* $g_{iQ} = 0$ *if* $\langle Q, \Lambda\rangle \neq 0$. *Here,* f_i, h_i *and* g_i *are series of class* $\mathscr{V}$ *in their respective variables.*

Proof. The proof of this theorem is completely analogous to the proof of the fundamental theorem on normal forms (theorem 2 in section 1.5 of this chapter). Just as we did there, we can obtain the following equation for the coefficients of $y_i Y^Q$:

$$\begin{aligned} g_{iQ} + h_{iQ}\langle Q, \Lambda\rangle = &- \sum_{P+R=Q} h_{iP} g_{iR} - \sum_{P+R=Q} h_{iP}\langle P, G_R\rangle \\ &+ \text{a sum of terms of form } f_{iS} h_{1P_1} \dots h_{1P_k} h_{2R_1} \dots h_{2R_l}\,, \end{aligned} \tag{12}$$

where

$$S + P_1 + \dots + P_k + R_1 + \dots + R_l = Q\ .$$

Let us denote by c_{iQ} the right-hand side of the i-th equation (12), so that

$$g_{iQ} + h_{iQ}\langle Q, \Lambda\rangle = c_{iQ}\ .$$

Now we order the vector indices P, Q, R in the following manner: P precedes Q if $\langle P, T\rangle < \langle Q, T\rangle$, where $T = -P^* - P_*$. Then for all f_{iS}, h_{iP}, and g_{iR}, we have $0 < \langle S, T\rangle$, $0 < \langle P, T\rangle$, and $0 < \langle R, T\rangle$. Therefore c_{iQ} depends only on those h_{jP} and g_{kR} for which vector indices P and R precede the vector Q in the above ordering. Moreover, there are only a finite number of integral vectors $P \in \mathrm{V}$ which precede Q. Hence, we can define the coefficients h_{iQ} and g_{iQ} just as we did in the principal theorem. □

Remark. If $\langle R_*, \Lambda\rangle = 0$, then the transformation (6) of the theorem can be chosen so that $h_{iQ} = 0$ for $\langle Q, \Lambda\rangle = 0$. In this case $g_{iQ} = f_{iQ}$ for $\langle Q, \Lambda\rangle = 0$. That is, the coefficients of the normal form (7) are found with no trouble: we simply retain those terms $f_{iQ}X^Q$ with exponents Q that lie on the corresponding edge of the sector **V**. The edge itself lies on the line **L** defined by the equation $\langle Q, \Lambda\rangle = 0$.

Proof of the Remark. If a point Q lies on a boundary of the sector **V** and $Q = P + R$, where P and $R \in \mathbf{V}$, then P and R must likewise lie on that boundary. This is a property of convex cones. Therefore, if Q lies on an edge of **V**, then all the vectors S, P, P_j, and R_j in equation (12) must lie on that edge. That is, all the coefficients h_{iP}, h_{1P_j}, and h_{2R_j} vanish, by the induction hypothesis. Consequently, equation (12) takes the form

$$g_{iQ} + h_{iQ}\langle Q, \Lambda\rangle = f_{iQ} \cdot$$

whence $g_{iQ} = f_{iQ}$ because $\langle Q, \Lambda\rangle = 0$; the h_{iQ} are arbitrarily set to zero. □

2.2. Integrating the Generalized Normal Form

We consider the case when system (1) is real. Then Λ is a real vector, and the equation

$$\langle Q, \Lambda\rangle = 0 \tag{12'}$$

defines a line **L** in the q_1, q_2 plane. The normalizing transformation (6) and normal form (7) will likewise have real coefficients. If the series of the given system and the normalizing transformation (6) converge, then we must examine the normal form in some set (11). The construction of the set $\mathscr{U}_{\mathbf{V}}(\varepsilon)$ for the situation in which R^* and R_* lie in the second and fourth quadrants, respectively, was discussed in § 3 of Chapter I. In what follows, we will assume that we are treating exactly that situation, and we will examine the integral curves of the normal form in the set $\mathscr{U}_{\mathbf{V}}(\varepsilon)$, which here consists of four curvilinear sectors. We will use the same classification of cases as earlier.

Case a) when the sole real solution of (12′) is the origin, is not allowed here; we proceed with case b).

b) The line **L** intersects the sector **V** only at the origin (fig. 61 $\mathbf{L} = \mathbf{L}(b)$). The normal form is the system

$$\dot{y}_i = \lambda_i y_i \ , \qquad i = 1, 2 \ ; \tag{13}$$

this integrates to $y_2 = c y_1^{\lambda_2/\lambda_1}$. We note here that

$$\frac{p_2^*}{p_1^*} = -\frac{r_1^*}{r_2^*} < \frac{\lambda_2}{\lambda_1} < \frac{p_{2*}}{p_{1*}} = -\frac{r_{1*}}{r_{2*}} \ .$$

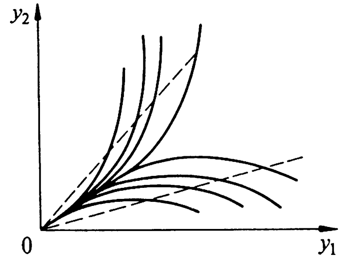

Fig. 62

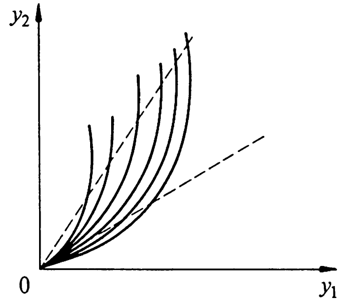

Fig. 63

That is, in the first quadrant of the y_1, y_2 plane the integral curves pass between the boundary curves, $y_2 = \varepsilon^{1/r_2^*} y_1^{-r_1^*/r_2^*}$ and $y_2 = \varepsilon^{1/r_{2*}} y_1^{-r_{1*}/r_{2*}}$, of the set $\mathscr{U}_{\mathbf{V}}(\varepsilon)$, as $y_1 \to 0$ (fig. 62, where the boundary curves are shown as dashed lines). The normalizing transformation in this case converges, and the behavior of the integral curves of the given system in each curvilinear sector of the set $\mathscr{U}_{\mathbf{V}}(\varepsilon)$ is the same: the integral curves enter the sector through both boundaries.

c) The line **L** lies along the boundary of the sector **V** (fig. 61, $\mathbf{L} = \mathbf{L}(c)$) and R_* is an integral vector with relatively prime components. The normal form is the system

$$\dot{y}_i = y_i \sum_{k=0}^{\infty} g_{ik}(Y^{R_*})^k = y_i g_i(Y^{R_*}) \ , \qquad i = 1, 2 \ .$$

Let $z = Y^{R_*}$ and $g(z) = r_{1*} g_1(z) + r_{2*} g_2(z)$. Then $\dot{z} = z g(z)$.

Now let us consider different sub-cases. If $g \equiv 0$, then the integral curves are $z = \text{const.}$, that is, $y_2 = c y_1^{-r_{1*}/r_{2*}}$; their behavior is pictured in figure 63. Here, the integral curves pass into the curvilinear sector of the set $\mathscr{U}_{\mathbf{V}}(\varepsilon)$ crossing the upper boundary, then approach the origin tangent to the lower boundary; some of the curves stay within the sector, while others exit across the lower boundary.

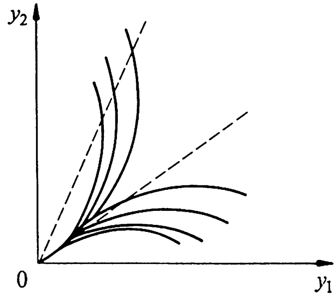

Fig. 64

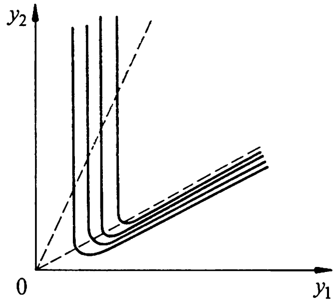

Fig. 65

Now let $g \not\equiv 0$, $g = az^k + \cdots$; then the solutions are

$$y_i = c_i \exp \int \frac{g_i\,dz}{zg(z)} = c_i \exp\left[-\frac{\lambda_i}{kaz^k}(1 + \cdots)\right], \qquad i = 1, 2\ ;$$

But how do these curves behave as $z \to 0$? If k is odd, then the curves will behave as in figure 64 in two of the four curvilinear sectors of the set $\mathscr{U}_{\mathrm{V}}(\varepsilon)$; in the other two sectors, they will be as pictured in figure 65. If k is even, then the integral curves will be similar in all four sectors: as in figure 64 if $\lambda_1/a > 0$, and as in figure 65 if $\lambda_1/a < 0$.

Thus, the behavior of the integral curves in the curvilinear sectors of the set $\mathscr{U}_{\mathrm{V}}(\varepsilon)$ is determined by the parity of the number k and the signs of the numbers a and λ_i. The formal normalizing transformation does not always converge here but, evidently, there always exists a smooth normalizing transformation. Therefore, the integral curves of the original system in the set $\mathscr{U}_{\mathrm{V}}(\varepsilon)$ will behave like those of the normal form.

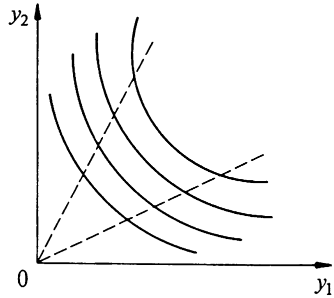

Fig. 66

d) The line **L** intersects the sector **V** at interior points. Here the normal form is system (13), or else takes a more complicated form. Again, the formal normalizing transformation may diverge, but there always exists an infinitely differentiable normalizing transformation (this follows from the results of Sternberg [1958, 1959] with the help of power transformations). The behavior of the integral curves in the curvilinear sectors of the set $\mathscr{U}_{\mathbf{V}}(\varepsilon)$ is pictured in figure 66. The solutions of the original system will behave similarly.

Thus, we have been able to determine the behavior of the integral curves of system (1) in the set $\mathscr{U}_{\mathbf{V}}(\varepsilon)$ in all cases, without any calculation whatsoever; all is determined directly by the coefficients of that system. In cases b) and d), it is sufficient to know λ_1 and λ_2, while in case c) it is also necessary to use those coefficients f_{iQ} for which the vectors Q lie along the corresponding side of the sector **V**.

Actual reduction to the normal form is only necessary if we seek more exact, analytic expressions for the solutions.

2.3. The Third Theorem on the Normal Form

Here we consider a situation which is in a sense intermediate between the situations of the principal theorem and the second theorem on normal forms. Let the right-hand sides of the equations in system (1) be series in integral, nonnegative powers of x_2, and let the set $\mathbf{D}(F)$ lie entirely within the sector

$$\mathbf{V} = \{Q\colon Q = \alpha_1 R^* + \alpha_2 R_*, \alpha_1 \geqslant 0, \alpha_2 \geqslant 0\} ,$$

where the vectors $R^* = (r_1^*, r_2^*)$ and $R_* = (r_{1*}, -1)$ are such that $r_1^* < 0 < r_2^*$, $r_{1*} > 0$, and $|r_1^*/r_2^*| < r_{1*}$. That is, the coefficient f_{1Q} in the series $f_1(X)$ vanishes unless the vector Q lies in the sector

$${}_1\mathbf{V} = \{Q\colon Q = \alpha_1 R^* + \alpha_2 E_1, \alpha_1 \geqslant 0, \alpha_2 \geqslant 0\} .$$

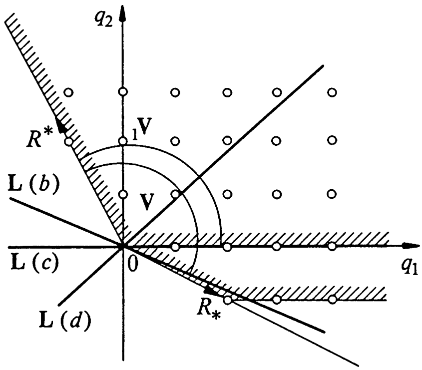

Fig. 67

We will denote the class of such power series f_1 by $_1\mathscr{V}(X)$. The coefficient f_{2Q} in the series $f_2(X)$ will vanish unless the vector Q lies either in the sector $_1\mathbf{V}$ or along the ray $q_2 = -1$, $q_1 \geqslant r_{1*}$ (figure 67). We will denote the class of such series by $_2\mathscr{V}(X)$. The set in X space corresponding to the sector $_1\mathbf{V}$ is

$$_1\mathscr{U}(\varepsilon, X) = \{X: |X|^{R^*} = \varepsilon, |x_1| \leqslant \varepsilon\}$$

(see figure 33). We will call the series $f_i \in {}_i\mathscr{V}(X)$ *convergent* if the series $\varphi_i = x_i f_i$ converge absolutely on some set $\mathscr{U}_{\mathbf{V}}(\varepsilon)$.

Note that the sector $_1\mathbf{V}$ is contained in the sector $\mathbf{V}$, but that the set $_1\mathscr{U}(\varepsilon)$ contains the set $\mathscr{U}_{\mathbf{V}}(\varepsilon)$.

Lemma 2. *If the series f_i of system* (1) *are convergent series of class $_i\mathscr{V}(X)$, and the series h_i in transformation* (6) *are convergent series of class $_i\mathscr{V}(Y)$, then in the system* (7), *obtained from system* (1), *the series g_i are convergent series of class $_i\mathscr{V}(Y)$ as well.*

Proof. We note that all series of class $_1\mathscr{V}(X)$ or $_2\mathscr{V}(X)$ also belong to the class $\mathscr{V}(X)$ corresponding to the sector $\mathbf{V}$. Recalling lemma 1, we realize that the series $g_i(Y)$ must be convergent series of class $\mathscr{V}(Y)$. It is evident from formula (10′) that the series $\psi_i(Y)$ contain only non-negative powers of the second coordinate, y_2, since the series φ_i and ξ_i have this property. Thus, the series ψ_i converge absolutely on set (11) and contain y_2 only in non-negative powers. But the first inequality in expression (11)

$$|Y|^{R_*} \leqslant \varepsilon \tag{14}$$

implies that $|y_1|^{r_{1*}}|y_2|^{-1} \leqslant \varepsilon$, or

$$|y_2| \geqslant \varepsilon^{-1}|y_1|^{r_{1*}}\,. \tag{15}$$

This places a lower bound on $|y_2|$. Let the series $\psi = \sum \psi_Q Y^Q$ contain y_2 only in non-negative powers, and let it converge absolutely at a point $y_1 = y_1^0$, $y_2 = y_2^0$. Then the series ψ converges absolutely at any point (y_1, y_2): $|y_1| = |y_1^0|$, $|y_2| \leqslant |y_2^0|$, since at such a point Y the series ψ is majorized by the convergent series $\sum |\psi_Q| |Y^0|^Q$. Consequently, if the series ψ converges absolutely along the curves $|y_2| = \varepsilon^{-1}|y_1|^{r_{1*}}$, then it converges for all Y satisfying $|y_2| \leqslant \varepsilon^{-1}|y_1|^{r_{1*}}$. That is, the region of absolute convergence of the series ψ_1 and ψ_2 is not limited by inequalities (14) and (15); it is limited only by the remaining inequalities in expression (11). Thus, the series ψ_i belong to the class ${}_i\mathscr{V}(Y)$ and converge absolutely on the set ${}_1\mathscr{U}(\varepsilon, Y)$. The lemma is proved. □

We proceed to consider formal series of class ${}_i\mathscr{V}$.

Theorem 2. *If the series f_i in a formal system* (1) *are of class ${}_i\mathscr{V}(X)$, then there exists a formal change of coordinates* (6), *where the h_i are series of class ${}_i\mathscr{V}(Y)$, which transforms system* (1) *into a system* (7) *in which the g_i are series of class ${}_i\mathscr{V}(Y)$ consisting only of resonant terms $g_{iQ} Y^Q$ with $\langle Q, \Lambda \rangle = 0$.*

The proof is analogous to the proof of the principal and second theorems on normal forms (see sections 1.5 and 2.1 of Chapter II), and is therefore left as an exercise for the reader.

Let us briefly consider the integrals of these normal forms on the sets ${}_1\mathscr{U}(\varepsilon, Y)$ for various cases:

b) The line **L** intersects the sector ${}_1$**V** only at the origin (figure 67, **L** = **L**(b)). The normal form has either the form of (13), or is $\dot{y}_1 = \lambda_1 y_1$, $\dot{y}_2 = \lambda_2 y_2 + a y_1^m$. In either case, the integral curves enter the curvilinear sectors of the set ${}_1\mathscr{U}(\varepsilon, Y)$ and, staying inside them, approach the origin (figure 68).

c) The line **L** is the q_1 axis. Here, just as in the situation of the principal theorem (section 1.5), we obtain four different kinds of behavior (figures 55–58).

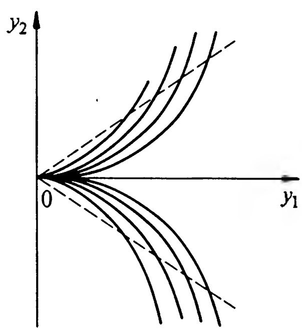

Fig. 68

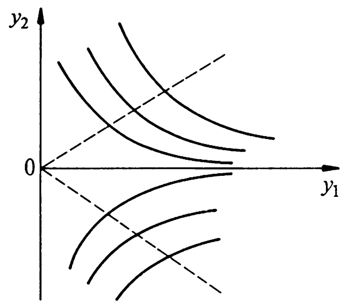

Fig. 69

d) The line **L** intersects the sector ${}_1\mathbf{V}$ at interior points. In this case, the integral curves behave as pictured in figure 69. Here, only one integral curve in each curvilinear sector of ${}_1\mathscr{U}(\varepsilon, \mathbf{Y})$ approaches the origin (as in the case of figure 58).

Remark. The principal theorem on normal forms is analogous to theorem 2 in § 3 of Chapter I, in the sense that each theorem gives a simplification of the original problem, using the linear terms. The second and third theorems on normal forms (sections 2.1 and 2.3, Chapter II) correspond to a vertex of the Newton open polygon. In that sense, they are analogous to theorem 5 of section 3.7 in Chapter I.

Exercises. 1. In case c) of section 2.2 for odd k, determine the disposition of the curvilinear sectors (like those in figures 64 and 65) in the various quadrants of the y_1, y_2 plane, depending on the parity of the coordinates of the vector R_* and the sign of the quotient λ_1/a.

2. In the situation of section 2.2, examine the case where the line **L** passes through R^*.

3. Prove theorem 2.

§3. A Nonelementary Singular Point

3.1. Statement of the Problem

For a system

$$\dot{x}_i = \varphi_i(x_1, x_2) , \qquad \varphi_i(0,0) = 0 , \qquad i = 1, 2 ,$$

let the functions φ_i be analytic at the origin, and let the origin be a non-elementary singular point. Then we can write the system as

$$\dot{x}_i = x_i f_i = x_i \sum f_{iQ} X^Q \tag{1}$$

and define the set

$$\mathbf{D} = \mathbf{D}(F) = \{Q: |f_{1Q}| + |f_{2Q}| \neq 0\} .$$

Clearly, $\mathbf{D} \subset \mathbf{N}$. We represent $\mathbf{D}$ as a set of points in the q_1, q_2 plane, and we find the convex Newton open polygon $\hat{\Gamma}$ which forms the boundary of the convex hull of the set $\mathbf{D}$ for orders $P < 0$. We denote the extreme vertices of $\hat{\Gamma}$ by Q_* and Q^*, as earlier. We index the vertices $\Gamma_j^{(0)}$ from bottom to top ($\Gamma_1^{(0)} = Q_*, \ldots, \Gamma_l^{(0)} = Q^*$), and likewise index the edges.

We present here a method of integrating such a system in a neighborhood of the origin. For each vertex $\Gamma_j^{(0)}$, we create a corresponding set $\mathcal{U}_j^{(0)}(\varepsilon)$. We assign the space $\mathcal{U}_j^{(1)}(\varepsilon)$ between the sets $\mathcal{U}_j^{(0)}(\varepsilon)$ and $\mathcal{U}_{j+1}^{(0)}(\varepsilon)$ to the edge $\Gamma_j^{(1)}$. Thus, the curvilinear sectors of the sets $\mathcal{U}_j^{(d)}(\varepsilon)$ entirely fill the neighborhood of the origin. In each of the sets $\mathcal{U}_j^{(0)}(\varepsilon)$, the system can be integrated by reducing it to a normal form (in general, the generalized normal form). The investigation of the system in the set $\mathcal{U}_j^{(1)}(\varepsilon)$ reduces with the help of a power transformation to studying a number of singular points of a second system. Each of these singular points is simpler than the original. Hence, after a finite number of steps the set $\mathcal{U}_j^{(1)}(\varepsilon)$ will be divided into a finite number of subsets, $\mathcal{U}_{jk}^{(1)}(\varepsilon)$, in each of which the system can be integrated with the help of a normal form. It then remains to "sew together" the results of integrating in the various regions $\mathcal{U}_{jk}^{(d)}(\varepsilon)$. Since we are interested only in the integral curves, in what follows we shall make various changes of the time variable.

We write system (1) in vector form:

$$(\ln^{\cdot} X) = \sum F_Q X^Q = F(X) \ .$$

Let $\Gamma_j^{(d)}$ be an edge or vertex of the open polygon $\hat{\Gamma}$ of this system. Let us call the function $\hat{F}_j^{(d)}(X) = \sum F_Q X^Q$, where $Q \in \Gamma_j^{(d)}$, the *truncation* of the vector-function F. Just as in the case of a scalar function f (see § 2, Chapter I), this truncation has its own cone $\mathbf{U}_j^{(d)}$. The only new element is the fact that the truncation $\hat{F} \neq 0$ as a vector, so that if just one component differs from zero, the other may vanish and the vector will still be said to be non-zero. For example, if $f_1 = x_1 + x_2$ and $f_2 = x_1^2 + x_2^2$, then the truncation for any vector order $P < 0$ is $\hat{F} = (\hat{f}_1, 0)$. Here, the second component of the vector $\hat{F}$ is not the truncation $\hat{f}_2$. We will henceforth denote the components of $\hat{F}$ by $\hat{f}_i$. For every truncated function $\hat{F}$ there is a corresponding *truncated system*

$$(\ln^{\cdot} X) = \hat{F}(X) \ . \qquad (1')$$

3.2. The Case of a Vertex

Let $\Gamma_j^{(0)} = \tilde{Q}$ be some vertex of the Newton open polygon $\hat{\Gamma}$ of system (1). Let $\Gamma_{j-1}^{(1)}$ and $\Gamma_j^{(1)}$ be the edges which intersect at $\Gamma_j^{(0)}$, and let R_{j-1} and R_j be unit vectors along those edges, with $r_{2,j-1}, r_{2,j} > 0$. We choose $R_* = -R_{j-1}$, $R^* = R_j$. If $\tilde{Q}$ is an extreme vertex of $\hat{\Gamma}$, for example $\tilde{Q} = Q_*$, then we choose $R_* = (1,0)$; if $\tilde{Q} = Q$, we choose $R^* = (0,1)$. We now divide system (1) by X^Q; that is, we introduce a new "time" t_1 such that $dt_1 = X^Q\, dt$. Then system (1) becomes the system

$$dx_i/dt_1 = x_i \sum_{Q \in \mathbf{D}} f_{iQ} X^{Q-\tilde{Q}} \ , \qquad i = 1, 2 \ . \qquad (2)$$

The exponents $Q - \tilde{Q}$ of this system are obtained from the exponents Q of system (1) by a parallel translation along the vector $\tilde{Q}$. Hence, the vertex $\Gamma_j^{(0)}$ of system (1) corresponds to the zero exponent in system (2), so that $\Lambda = F_{\tilde{Q}}$. All the vector exponents $Q - \tilde{Q}$ lie in the sector $\mathbf{V}$ which is bounded by half-lines along the vectors R_* and R^* (figure 61). Therefore, system (2) satisfies the conditions of the second theorem on the normal form (section 2.1 of Chapter II). According to that theorem, there exists on the set

$$\mathcal{U}_j^{(0)}(\varepsilon) = \{X \colon |X|^{R^*} \leqslant \varepsilon, |X|^{R_*} \leqslant \varepsilon\} \qquad (2')$$

a coordinate transformation

$$x_i = y_i + \xi_i(Y) \ , \qquad i = 1, 2 \ , \qquad (3)$$

under which system (2) is transformed into the normal form

$$dy_i/dt_1 = y_i \sum_{\langle Q, \Lambda \rangle = 0} g_{iQ} Y^Q \ , \qquad i = 1, 2 \ . \qquad (4)$$

As we showed in § 2 of this chapter, this normal form is integrable. The solutions of the normal form (4) yield, through transformation (3), the solutions of system (2) in the set (2′). Since the integral curves of systems (1) and (2) coincide, it is possible by this method to find the integral curves (and solutions) of system (1) in the set (2′) with any degree of accuracy. If we are interested only in the behavior of the integral curves in a neighborhood of the origin, then it is sufficient to find the character of that behavior in the set (2′). As shown in § 2 of this chapter, it is unnecessary to do any calculations for this purpose—simply knowing the coefficients of system (1) is sufficient. All of this is a general construction. For a system (1) with analytic right-hand sides, there are some specifics which simplify the analysis of vertices as well as the general picture. We will identify four types of vertices $\tilde{Q}$.

Type I. The vertex $\tilde{Q}$ is one of the extreme vertices, Q_* or Q^*, and lies outside of the first quadrant. Then one of the coordinates of $\tilde{Q}$ is -1. Suppose that $q_2 = -1$; then the point $\tilde{Q}$ belongs to the set $\mathbf{N}_2$ but not to the set $\mathbf{N}_1$. Therefore the coefficient $f_{1\tilde{Q}}$ in system (1) is zero. The vertex $\tilde{Q} = Q_*$ lies on the lower horizontal support line of the set $\mathbf{D}(F)$, so that $R_* = (1, 0)$, and the corresponding set $\mathscr{U}_j^{(0)}(\varepsilon)$ is

$$\mathscr{U}_*(\varepsilon) = \{X : |X|^{R^*} \leqslant \varepsilon, |x_1| \leqslant \varepsilon\} . \tag{5}$$

This set has the same form as is pictured in figure 33. Since $\Lambda = F_{\tilde{Q}} = (0, f_{2\tilde{Q}})$, the solutions of the equation $\langle Q, \Lambda \rangle = 0$ in the sector $\mathbf{V}$ are $(k, 0)$. Since $\langle R_*, \Lambda \rangle = 0$, both our second theorem on the normal form (section 2.1) and the remarks about that theorem apply. According to the remark, the normal form (4) has the form

$$dy_i/dt_1 = y_i \sum_{k=0}^{\infty} g_{i(k,0)} Y^{(k,0)} = y_i \sum f_{i(k,0)+\tilde{Q}} y_1^k , \qquad i = 1, 2 . \tag{6}$$

But here $f_{1(k,0)+\tilde{Q}} = 0$ since the second coordinate of each vector index $(k, 0) + \tilde{Q}$ is equal to -1; that is, these vector indices do not lie in the set $\mathbf{N}_1$. Consequently, the first equation of system (6) is $dy_1/dt_1 = 0$, so the lines $y_1 = \text{const.}$ are integral curves of the normal form (6). Hence, in the set (5) corresponding to this first type of vertex, the integral curves will behave as they do in figure 33; that is, in the set $\mathscr{U}_*(\varepsilon)$, there is no integral curve which passes through the origin.

Type II. The vertex $\tilde{Q}$ is one of the extreme vertices Q_* or Q^*, and $\tilde{Q} \geqslant 0$. Suppose that $\tilde{Q} = Q_* \geqslant 0$. Then the vertex $\tilde{Q}$ corresponds to a sector $\mathbf{V}$ in the q_1, q_2 plane, bounded by rays collinear with the vectors $R^* = (r_1^*, r_2^*)$ and $R_* = (1, 0)$, where $r_1^* < 0 < r_2^*$. For each point $Q \in \mathbf{D}(F)$, the difference $Q - \tilde{Q}$ lies in $\mathbf{V}$. The vector $\mathbf{V}$ corresponds to the set (5) in the x_1, x_2 plane. The second theorem on the normal form applies to system (2); it implies that the x_1 axis is an integral curve. The behavior of the integral curves in the rest of set (5) is like that in figure 68, figure 69, or figure 55. All of the cases enumerated at the end of section 2.3 are possible here.

Type III. $\tilde{Q}$ is not an extreme vertex, but one of its coordinates is zero. Suppose that $q_2 = 0$. In our general construction, this vertex corresponds to a sector **V** and a set (2′). But, by the construction of section 2.3, we can create the sector

$$ {}_1\mathbf{V} = \{Q: Q = \alpha_1 R_1^* + \alpha_2 E_1; \alpha_1, \alpha_2 \geqslant 0\} \tag{7}$$

and a set

$$ {}_1\mathscr{U}(\varepsilon) = \{X: |X|^{R^*} \leqslant \varepsilon, |x_1| \leqslant \varepsilon\} \ . \tag{7'}$$

corresponding to such a vertex. Here

$$\mathbf{V} \supset {}_1\mathbf{V} \ , \qquad \mathscr{U}_{\mathbf{V}}(\varepsilon) \subset {}_1\mathscr{U}(\varepsilon) \ .$$

The construction of set (7′) is similar to that of set (5). Our third theorem on the normal form (section 2.3) applies to system (2) within set (7′). At the end of § 2, we showed that in any curvilinear sector of the set (7′) there must be at least one integral curve which passes through the origin (in case (c), it may consist of fixed points). In the situation under consideration, the vertex $\tilde{Q}$ is $\Gamma_2^{(0)}$; to its right on the Newton open polygon are the edge $\Gamma_1^{(1)}$ and the vertex $\Gamma_1^{(0)} = Q_*$. We note here that set (7′) includes both $\mathscr{U}_1^{(1)}(\varepsilon)$ and $\mathscr{U}_1^{(0)}(\varepsilon)$, the sets corresponding to $\Gamma_1^{(1)}$ and $\Gamma_1^{(0)}$. Without the third theorem on the normal form, we should have to study system (1) in three different sets $\mathscr{U}_1^{(0)}(\varepsilon)$, $\mathscr{U}_1^{(1)}(\varepsilon)$, and $\mathscr{U}_2^{(0)}(\varepsilon)$. The third theorem allows us to examine the system in set (7′), which includes all three of the sets named. That is, the integration of the system in the set (7′) corresponding to the vertex $\tilde{Q}$ includes integration in the set $\mathscr{U}_1^{(0)}(\varepsilon)$ corresponding to the vertex $Q_* = \Gamma_1^{(0)}$. Therefore, it is possible to treat a vertex of the first type without creating a separate set $\mathscr{U}_1^{(0)}(\varepsilon)$ and integrating the system there, as long as it is adjacent to a vertex of type III.

Type IV. $\tilde{Q} > 0$, and $\tilde{Q}$ is not an extreme vertex. For such a vertex, there is no way to simplify the application of the second theorem on the normal form. Here, the set $\mathscr{U}_j^{(0)}(\varepsilon)$ consists of four curvilinear sectors, as in figure 31. In the first quadrant of the x_1, x_2 plane, the sector corresponding to one vertex lies above the sector corresponding to a lower vertex (see § 3, Ch. I). As remarked in § 2 of this chapter, the behavior of the integral curves of system (1) in the sectors of the set $\mathscr{U}_j^{(0)}(\varepsilon)$ can be determined from the coefficients F_Q with no further calculations necessary.

3.3. The Case of an Edge

Let $\Gamma_j^{(1)}$ be some edge of the Newton open polygon of system (1). The endpoints of this edge are the vertices $\Gamma_j^{(0)}$ and $\Gamma_{j+1}^{(0)}$, and the unit vector along the edge is $R = R_j = (\Gamma_{j+1}^{(0)} - \Gamma_j^{(0)})/m$. The sets $\mathscr{U}_j^{(0)}(\varepsilon)$ and $\mathscr{U}_{j+1}^{(0)}(\varepsilon)$, corresponding to the vertices $\Gamma_j^{(0)}$ and $\Gamma_{j+1}^{(0)}$, are restricted by the inequalities $|X|^R \leqslant \varepsilon$ and $|X|^{-R} \leqslant \varepsilon$. Hence, for a set corresponding to the edge $\Gamma_j^{(1)}$, we are left with

$$\mathscr{U}_j^{(1)}(\varepsilon) = \{X: \varepsilon \leqslant |X|^{R_j} \leqslant \varepsilon^{-1}, |x_1| \leqslant \varepsilon, |x_2| \leqslant \varepsilon\} \ . \tag{8}$$

Just as we did in § 3 of Chapter I, we investigate the behavior of system (1) in set (8) by applying a power transformation

$$y_1 = x_1^{s_1} x_2^{s_2} , \qquad \alpha = \begin{pmatrix} s_1 & s_2 \\ r_1 & r_2 \end{pmatrix} \tag{9}$$
$$y_2 = x_1^{r_1} x_2^{r_2} ,$$

where the matrix α has integral elements and $\det \alpha = 1$. We note that if $X \to 0$ in the set $\mathscr{U}_j^{(1)}(\varepsilon)$, then $y_2 \to \text{const.} \neq 0$ and $y_1 \to 0$. In fact, $s_1 r_2 = 1 + s_2 r_1$, so

$$y_1^{r_2} = x_1^{s_1 r_2} x_2^{s_2 r_2} = x_1 (x_1^{r_1} x_2^{r_2})^{s_2} = x_1 y_2^{s_2} .$$

Consequently, $y_1^{r_2} \to 0$ as $x_1 \to 0$, since $y_2 \to \text{const} \neq 0, \infty$; recall that $r_2 > 0$. Thus, part of the set $\mathscr{U}_j^{(1)}(\varepsilon)$ near the origin is transformed into a part of a neighborhood of the y_2 axis:

$$\mathscr{U}_j^{(1)}(\varepsilon) = \{Y: |y_1| \leqslant \varepsilon \leqslant |y_2| \leqslant \varepsilon^{-1}\} . \tag{10}$$

Under this power transformation, system (1) becomes

$$(\ln \dot{Y}) = \sum_{Q' \in \mathbf{D}'} F'_{Q'} Y^{Q'} \equiv F'(Y) , \tag{11}$$

the open polygon $\hat{\Gamma}$ of system (1) becomes the polygon $\hat{\Gamma}'$ of system (11), and the edge $\Gamma_j^{(1)}$ becomes a vertical edge $\Gamma_j^{(1)'}$. The truncated system (1′), corresponding to the edge $\Gamma_j^{(1)}$, is transformed into a truncated system

$$(\ln \dot{Y}) = \sum_{Q' \in \mathbf{D}' \cap \Gamma_j^{(1)'}} F'_{Q'} Y^{Q'} \equiv \hat{F}'(Y) , \tag{12}$$

corresponding to the edge $\Gamma_j^{(1)'}$ (fig. 16). Since $\Gamma_j^{(1)'}$ is a vertical edge, its points Q have a common coordinate $q'_1 = r$; i.e., $\hat{F}'(Y) = y_1^r \hat{F}_0'(y_2)$. We now divide system (11) by the maximum power y_1^s such that we obtain a system

$$dy_i/dt_1 = y_1^{-s} y_i f_i'(Y) = \zeta_i(Y) , \qquad i = 1, 2 , \tag{13}$$

where the right-hand sides are expressed only in non-negative powers of y_1. Here, $dt_1 = y_1{}^s dt$.

Thus, we arrive at the following **problem**: Investigate the behavior of the integral curves of system (13) in set (8), recalling that the y_2 axis corresponds to the point $x_1 = x_2 = 0$. In order to solve this problem, we need to clarify the character of those integral curves which pass through the finite points of the y_2 axis but differ from $y_2 = 0$. It remains to be noted that after division by y_1^s, truncation (12) of system (11) becomes the truncated system

$$dy_i/dt_1 = y_1^{r-s} y_i \hat{f}_{i0}'(y_2) , \qquad i = 1, 2 \tag{14}$$

of system (13). We can identify two types of such truncations (14).

Type I. The second component of the vector $\hat{F}'(Y)$ is $\hat{f}_2'(Y) \not\equiv 0$. In this case, $r = s$, and the y_2 axis is an integral curve, along which $\zeta_1(0, y_2) \equiv 0$. The points of this axis, at which $\zeta_2(0, y_2) \neq 0$, are simple. In accordance with theorem 1 (§ 1, Ch. II), the integral curves in the neighborhood of such a simple point are lines parallel to the y_2 axis. If we consider the fact that the y_2 axis corresponds to a single point, $x_1 = x_2 = 0$, then it is clear that the corresponding integral curves of system (1) cannot approach the singular point. Now we are left with the singular points on the y_2 axis ($y_2 \neq 0$, $y_2 \neq \infty$).

Such singular points, say $y_1 = 0$, $y_2 = y_2^0$, are obtained as roots $y_2 = y_2^0$ of the equation $\hat{f}_{20}'(y_2) = 0$. If at such a point $d\hat{f}_{20}'/dy_2 \neq 0$ or $\hat{f}_{10}' \neq 0$, then it is an elementary singular point of system (13), and its neighborhood can be investigated by transforming the system into a normal form. If, however, $d\hat{f}_{20}'/dy_2 = 0$ and $\hat{f}_{10}' = 0$, then the point is a non-elementary singularity, and to determine its character one must proceed to construct a Newton open polygon, and so on. However, it is now a simpler singularity than the original.

Type II. If $\hat{f}_2'(Y) \equiv 0$, then $\hat{f}_1' \not\equiv 0$, and $s = r + 1$. In this case the y_2 axis is not an integral curve of (13). If $\hat{f}_{10}'(y_2^0) \neq 0$, then the point $(0, y_2^0)$ is a simple point of system (13); one integral curve, which is not tangent to the y_2 axis, passes through this point. In the x_1, x_2 plane, the corresponding integral curve passes through the singular point $x_1 = x_2 = 0$. If $\hat{f}_{10}'(y_2^0) = 0$ and $\zeta_2(0, y_2^0) \neq 0$, then the point $(0, y_2^0)$ is a simple point of system (13), but the integral curve through the point is tangent to the y_2 axis there. Along this curve, y_1 can be expressed in integral powers of $y_2 - y_2^0$ (Cauchy's Theorem), and we can investigate this solution. If $\hat{f}_{10}'(y_2^0) = 0$ and $\zeta_2(0, y_2^0) = 0$, then $(0, y_2^0)$ is a singular point of system (13). If it is an elementary singularity, the system is integrable in a neighborhood of the point; if it is non-elementary, it is necessary to construct a Newton polygon, divide the neighborhood into sets $\mathcal{U}'(\varepsilon)$, and so forth.

Thus, for both types, we must find at the y_2 axis the real roots y_2^0 of one of the algebraic equations:

$$\hat{f}_{20}'(y_2) = 0 \quad \text{for type I} \quad \text{and}$$

$$\hat{f}_{10}'(y_2) = 0 \quad \text{for type II} \ .$$

There is a finite number of such roots: $y_2^{(1)}, \ldots, y_2^{(m)}$. Each point $y_1 = 0$, $y_2 = y_2^{(k)}$ has its own neighborhood $\mathcal{U}_k'(\varepsilon)$ in the y_1, y_2 plane. The remainder of set (10) is divided into a finite number of regions $\tilde{\mathcal{U}}_l'(\varepsilon)$ which contain no singular points. The system is integrable in each of these regions using theorem 1 of § 1 in Chapter II. In the neighborhood $\mathcal{U}_k'(\varepsilon)$, the system (13) is integrable either immediately or through further division into sets $\mathcal{U}_{kl}'(\varepsilon)$. The division of set (10) into the parts $\mathcal{U}_k'(\varepsilon)$ and $\tilde{\mathcal{U}}_l'(\varepsilon)$ corresponds, in the x_1, x_2 plane, to dividing the set $\mathcal{U}_j^{(1)}(\varepsilon)$ into more refined sets $\mathcal{U}_{jk}^{(1)}(\varepsilon)$ and $\tilde{\mathcal{U}}_{jl}^{(1)}(\varepsilon)$. Moreover, system (13) (and, consequently, (1)) is immediately integrable in the sets $\tilde{\mathcal{U}}_{jl}^{(1)}(\varepsilon)$, while in the sets $\mathcal{U}_{jk}^{(1)}(\varepsilon)$ further

refinement may be necessary. This depends on the complexity of the singularity. Generally speaking, this process of successive fragmentation of the sets $\mathcal{U}(\varepsilon)$ and resolving singularities may require many steps, but it will always provide a solution in a finite number of steps. This can be proved for differential equations exactly as it was for analytic equations in section 2.9 of Chapter I, using our concept of the "height" of edges and singularities.

Remark. The functions $\hat{f}_{10}'(y_2)$ and $\hat{f}_{20}'(y_2)$ not only determine the distribution of secondary singularities, but also, in the case of an elementary singularity of the first type, yield the eigenvalues $\lambda_1 = \hat{f}_{10}'(y_2^0)$ and $\lambda_2 = d\hat{f}_{20}'(y_2^0)/dy_2$. Thus, in many (but not all) cases, these functions entirely determine the character of the behavior of the integral curves. But these functions $\hat{f}_{10}'(y_2)$ and $\hat{f}_{20}'(y_2)$ are obtained from the truncated system (12), which itself is derived from the truncated system (1′) with a coordinate change (9). Consequently, many properties of the solutions of system (1) can be discovered from the truncated system (1′). Similar investigations of an n-dimensional system have been made by Shestakov [1960, 1961], Bruno [1965], and Beklemisheva [1972].

A second practical method which facilitates calculation is limited to cases when the edge $\Gamma_j^{(1)}$ lies entirely outside the first quadrant of the q_1, q_2 plane. We need not examine it here since a vertex $\tilde{Q}$ of this edge must lie on the boundary of the first quadrant, and is thus a vertex of type III; the set (7′) corresponding to that vertex entirely contains the set $\mathcal{U}_j^{(1)}(\varepsilon)$ corresponding to the edge $\Gamma_j^{(1)}$, so that the last set need not be treated separately.

3.4. Synthesis

Thus, a neighborhood of the non-elementary singularity $X = 0$ of system (1) is divided into a finite number of sets $\mathcal{U}_{ijk\ldots l}(\varepsilon)$. In each such set, we introduce its own coordinates y_1, y_2 in which the system is integrable. Returning from Y to X coordinates, we obtain parts of the solutions of system (1) in each set $\mathcal{U}_{ijk\ldots l}(\varepsilon)$ separately. Once these parts of solutions are "sewn together," we will have integrated the system in the entire neighborhood. But a new difficulty arises here. The transformations from Y to X coordinates employ infinite series, so that these transformations can, in reality, only be calculated approximately (i.e., to terms of some finite degree). As a result, the parts of the solutions X in each of the sets $\mathcal{U}_{ijk\ldots l}(\varepsilon)$ will only be approximate. And even though these parts of the solutions can be obtained arbitrarily accurately, the complexity of the calculations increases with increasing accuracy. To sew together the parts of the solutions in neighboring parts of the sets $\mathcal{U}_{ijk\ldots l}(\varepsilon)$, it is necessary that the approximate solutions be sufficiently close to the actual solutions. This requires different amounts of calculation in different cases.

Let us consider the problem of clarifying the topological portrait of the integral curves in the neighborhood of a non-elementary singular point. We identify two cases:

1) Among all the sets $\mathscr{U}_{ijk\ldots l}(\varepsilon)$ there is at least one in which an integral curve approaches the singularity. In this case, it is sufficient to divide the neighborhood $\mathscr{U}$ into sets $\mathscr{U}_{ijk\ldots l}(\varepsilon)$ and to integrate system (1) in each of those sets in the appropriate Y coordinates. Sewing together the neighboring parts of the solutions with topological accuracy presents no problem.

2) Not one of the sets $\mathscr{U}_{ijk\ldots l}(\varepsilon)$ contains an integral curve which approaches the origin inside that set. Instead, the integral curves pass from one set $\mathscr{U}_{ijk\ldots l}(\varepsilon)$ to another, and never stay in any one. In this case, the integral curves simply twist around the singularity, either as closed curves (in the case of a center) or as spirals (in the case of a focus). The problem of distinguishing between a center and a focus is, generally speaking, extremely complicated. We leave it for § 4 of this chapter.

3.5. Examples 1, 2, 3

We now consider a few examples of the first case.

Example 1. We consider the system

$$\dot{x}_1 = x_2 + x_1^2 - x_1 x_2 = x_1(x_1^{-1}x_2 + x_1 - x_2) \ ,$$

$$\dot{x}_2 = ax_1^2 + 3x_1x_2 - 5x_2^2 = x_2(ax_1^2x_2^{-1} + 3x_1 - 5x_2) \ ,$$

where the parameter a is non-zero. Here,

$$Q_1 = (-1, 1) \ , \quad Q_2 = (1, 0) \ , \quad Q_3 = (0, 1) \ , \quad Q_4 = (2, -1) \ ;$$

$$F_{Q_1} = (1, 0) \ , \quad F_{Q_2} = (1, 3) \ , \quad F_{Q_3} = (-1, -5) \ , \quad F_{Q_4} = (0, a) \ .$$

The Newton open polygon $\hat{\Gamma}$ consists of a single edge $\Gamma_1^{(1)}$ and the two vertices $\Gamma_1^{(0)} = Q_4$ and $\Gamma_2^{(0)} = Q_1$ (fig. 70). Both of these vertices are of type I: the disposition of the sets $\mathscr{U}_1^{(0)}(\varepsilon)$ and $\mathscr{U}_2^{(0)}(\varepsilon)$ and the behavior of integral curves in them is roughly sketched in figure 71. The remaining set $\mathscr{U}_1^{(1)}(\varepsilon)$ is restricted by

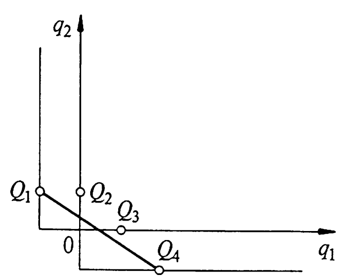

Fig. 70

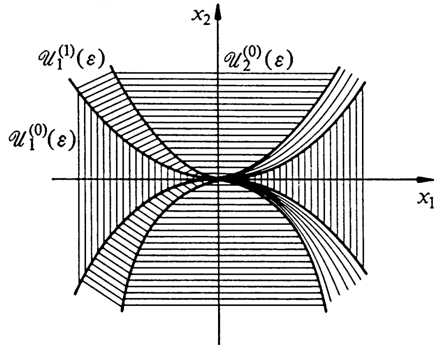

Fig. 71

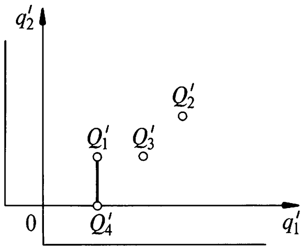

Fig. 72

the inequalities $\varepsilon \leq |X|^R \leq \varepsilon^{-1}$, where $R = Q_1 - Q_4 = (-3, 2)$ is the unit vector along the edge $\Gamma_1^{(1)}$. If we seek a vector $S = (s_1, s_2)$ such that $s_1 r_2 - s_2 r_1 = 1$, we can choose $S = (2, -1)$. We then perform the power transformation

$$y_1 = x_1^2 x_2^{-1} , \qquad y_2 = x_1^{-3} x_2^2 ;$$

whose inverse is

$$x_1 = y_1^2 y_2 , \qquad x_2 = y_1^3 y_2^2 .$$

The original system becomes

$$(\ln \dot{y}_1) = 2y_1 y_2 - y_1^2 y_2 + 3y_1^3 y_2^2 - ay_1 ,$$

$$(\ln \dot{y}_2) = -3y_1 y_2 + 3y_1^2 y_2 - 7y_1^3 y_2^2 + 2ay_1 ;$$

the second component of the truncated system, $\hat{f}_2' = -3y_1 y_2 + 2ay_1$, is not identically zero (figure 72). Consequently, it is of type I by the classification of section 3.3. Dividing by y_1, we obtain the system

$$dy_1/dt_1 = 2y_1y_2 - y_1^2y_2 + 3y_1^3y_2^2 - ay_1 \ ,$$

$$dy_2/dt_1 = -3y_2^2 + 3y_1y_2^2 - 7y_1^2y_2^3 + 2ay_2 \ .$$

Here, the y_2 axis is an integral curve; the only singular point on it is $y_1 = 0$, $y_2^0 = 2a/3$, obtained from the equation

$$\hat{f}'_{20}(y_2) = -3y_2 + 2a = 0 \ .$$

The only integral curve passing through all remaining points $y_1 = 0$, $y_2 \neq y_2^0, 0$, is the y_2 axis itself. To investigate the neighborhood of the singular point $Y = (0, 2a/3)$, we make a parallel translation

$$y_2 = y_2^0 + z_2 = \tfrac{2}{3}a + z_2 \ ,$$

and obtain

$$dy_1/dt_1 = \tfrac{1}{3}ay_1 + 2y_1z_2 - y_1^2(\tfrac{2}{3}a + z_2) + 3y_1^3(\tfrac{2}{3}a + z_2)^2 \ ,$$

$$\begin{aligned} dz_2/dt_1 &= -2az_2 - 3z_2^2 + 3y_1(\tfrac{2}{3}a + z_2)^2 - 7y_1^2(\tfrac{2}{3}a + z_2)^3 \\ &= -2az_2 + \tfrac{4}{3}a^2y_1 + 4ay_1z_2 + 3y_1z_2^2 - 3z_2^2 - 7y_1^2(\tfrac{2}{3}a + z_2)^3 \ . \end{aligned}$$

The matrix of coefficients of linear terms of this system is

$$A = \begin{pmatrix} \dfrac{a}{3} & 0 \\ \dfrac{4}{3}a^2 & -2a \end{pmatrix} .$$

and its eigenvalues are $\lambda_1 = a/3$ and $\lambda_2 = -2a$. For any real $a \neq 0$, we find that $\lambda = \lambda_1/\lambda_2 = -1/6$. The singular point is a saddle, and exactly two integral curves pass through it. One is the z_2 axis, and the other is tangent to the eigenvector B_1 of the matrix A. We find that $B_1 = (7, 4a)$ from the equation $AB_1 = \lambda_1 B_1$; that is, B_1 lies along the line $z_2 = 4ay_1/7$. Thus, the only integral curve which crosses the y_2 axis is $z_2 = 4ay_1/7 + \cdots$, or $y_2 = (2/3)a + 4ay_1/7 + \cdots$ (fig. 73). This curve has the two half branches $\mathscr{F}_1'(y_1 > 0)$ and $\mathscr{F}_2'(y_1 < 0)$. In the original coordinates, we have

$$x_1 = \tfrac{2}{3}ay_1^2 + \tfrac{4}{7}ay_1^3 + \cdots \ , \qquad x_2 = \tfrac{4}{9}a^2y_1^3 + \tfrac{16}{21}ay_1^4 + \cdots \ .$$

This is the only integral curve which passes through the origin $x_1 = x_2 = 0$; it consists of two half branches $\mathscr{F}_1$ and $\mathscr{F}_2$ (shown for $a > 0$ in figs. 71 and 74). The remaining integral curves behave as shown by figure 71. Thus, the general picture will be as sketched in figure 74 (for $a > 0$).

Note that the truncated system corresponding to the edge $\Gamma_1^{(0)}$ is sufficient to construct figure 74. In fact, the power transformation and the values of y_2^0, λ_1,

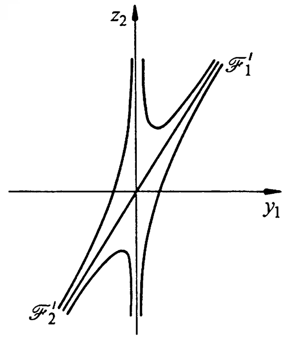

Fig. 73

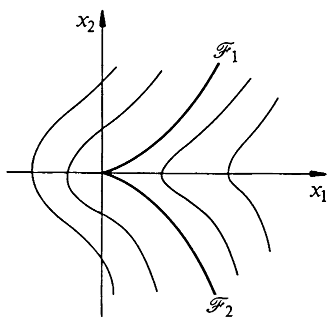

Fig. 74

and λ_2 were all obtained from the coefficients of the truncated system. Only the calculation of the first term in the expression of z_2 in terms of y_1 along the half branches $\mathscr{F}_1$ and $\mathscr{F}_2$ required us to use terms of the original system that were not included in the truncated system. And this gave us the second terms in the expression of x_1 and x_2 in terms of y_1 along the half branches $\mathscr{F}_1$ and $\mathscr{F}_2$. In our case, these second terms are not important for sketching the picture; the behavior of the half branches $\mathscr{F}_1$ and $\mathscr{F}_2$ is entirely determined by the first terms

$$x_1 = \tfrac{2}{3}ay_1^2 + \cdots, \qquad x_2 = \tfrac{4}{9}a^2y_1^3 + \cdots$$

Thus, in this example it would have been sufficient, for the purpose of finding the behavior of the integral curves, to perform all of our calculations (i.e. the power transformation and finding the root y_2^0) with the truncated system. This would significantly simplify computations. Therefore the following method of calculation is recommended for investigating the set $\mathscr{U}_j^{(1)}(\varepsilon)$ corresponding to an

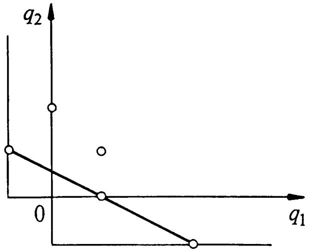

Fig. 75

edge $\Gamma_j^{(1)}$. Instead of the original system, take the corresponding truncated system and apply to it the necessary transformations. Only if it is impossible to determine the character of the solutions from the coefficients of the truncated system (i.e., in case of an edge of type II, and also an edge of type I if $\lambda_1 = 0$ at the singular point on the y_2 axis) should those transformations be carried out with the remaining terms of the original system.

Example 2.

$$\dot{x}_1 = x_2 + ax_1^2 + 5x_1x_2^2 = x_1(x_1^{-1}x_2 + ax_1 + 5x_2^2) \ ,$$

$$\dot{x}_2 = -2x_1^3 + 3x_1x_2^2 - 4x_2^3 = x_2(-2x_1^3x_2^{-1} + 3x_1x_2 - 4x_2^2) \ .$$

The open polygon $\hat{\Gamma}$ (fig. 75) consists of one edge and two vertices. Both vertices are of the first type, so that the disposition of the sets $\mathscr{U}_j^{(0)}(\varepsilon)$ and integral curves will be just as in example 1. The unit vector along the edge is $R = (-2, 1)$ and the truncated system is

$$\dot{x}_1 = x_1(x_1^{-1}x_2 + ax_1) \ , \qquad \dot{x}_2 = -2x_1^3 \ .$$

We choose $S = (1, 0)$ for the given R and introduce new coordinates $y_1 = x_1$, $y_2 = x_1^{-2}x_2$; the inverse transformation is $x_1 = y_1$, $x_2 = y_1^2y_2$. The truncated system becomes

$$\dot{y}_1 = y_1^2y_2 + ay_1^2 \ , \qquad \dot{y}_2 = -2y_1 - 2ay_1y_2 - 2y_1y_2^2 \ .$$

Letting $dt_1 = y_1\,dt$, we obtain

$$dy_1/dt_1 = y_1y_2 + ay_1 \equiv g_1y_1 \ , \qquad dy_2/dt_1 = -2 - 2ay_2 - 2y_2^2 \equiv g_2 \ .$$

To find the singular points along the y_2 axis we obtain the equation

$$-\tfrac{1}{2}g_2 = y_2^2 + ay_2 + 1 = 0 \ ,$$

whence we obtain

$$y_2^0 = -\frac{a}{2} \pm \sqrt{\frac{a^2}{4} - 1} \ .$$

If $|a| < 2$, then there are no real roots and none of the integral curves in the set $\mathscr{U}_1^{(1)}(\varepsilon)$ approaches the singular point. Since no such curves exist in the sets $\mathscr{U}_j^{(0)}(\varepsilon)$, we are faced with the problem of distinguishing between a center and a focus. That problem can be solved for this system, but we will not do so here. If $|a| > 2$, there are two real roots. For the sake of discussion, let us choose $a = -4$. Then $y_2^0 = 2 \pm \sqrt{3}$. The eigenvalues at these points are

$$\lambda_1 = g_1(y_2^0) = a + y_2^0 = -2 \pm \sqrt{3} < 0 \ ,$$

$$\lambda_2 = dg_2(y_2^0)/dy_2 = -2(a + 2y_2^0) = \mp 4\sqrt{3} \ .$$

Therefore, for the root $y_2^0 = 2 + \sqrt{3}$, we have $\lambda = \lambda_1/\lambda_2 > 0$, and the singular point $y_1 = 0$, $y_2^0 = 2 + \sqrt{3}$ is a node. For the other root, $y_2^0 = 2 - \sqrt{3}$, we have $\lambda < 0$, and the singular point $y_1 = 0$, $y_2 = 2 - \sqrt{3}$ is a saddle. The behavior of the integral curves is sketched in figure 76. Returning to the original variables, we obtain figure 77.

If $|a| = 2$, the roots coincide. Suppose that $a = -2$; then $y_2^0 = 1$, $\lambda_1 = -1$, $\lambda_2 = 0$. We let $z_2 = y_2 - y_2^0 = y_2 - 1$. To investigate the system in the entire neighborhood of $y_1 = 0$, $z_2 = 0$, we must find the normal form. In the complete system

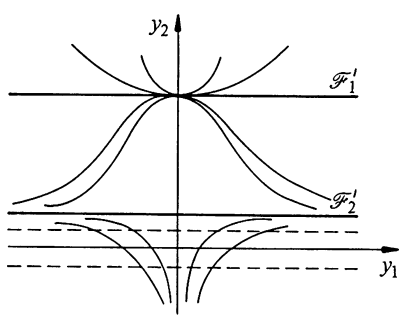

Fig. 76

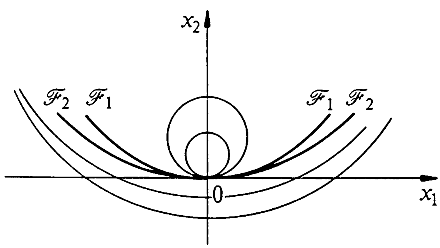

Fig. 77

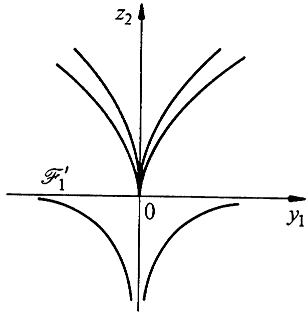

Fig. 78

$$dy_1/dt_1 = y_1 \sum h_{1Q} y_1^{q_1} z_2^{q_2} \ , \qquad dz_2/dt_1 = z_2 \sum h_{2Q} y_1^{q_1} z_2^{q_2}$$

all terms have $q_1 \geqslant 0$, and all of the terms with $q_1 = 0$ are obtained from the truncated system. On the other hand, $\Lambda = (-1, 0)$, so that the normal form

$$dw_i/dt_1 = w_i \sum g_{iQ} W^Q \ , \qquad i = 1, 2$$

will only have terms with $q_1 = 0$. But according to the remark made after the second theorem on the normal form, we will have $g_{i(0,q_2)} = h_{i(0,q_2)}$ here, so that the normal form is, in fact, just the truncated system. Thus, investigation of the full system in the neighborhood of the singular point $y_1 = 0$, $y_2 = 1$ reduces to considering the truncated system

$$dy_1/dt_1 = -y_1 + y_1 z_2 \ , \qquad dz_2/dt_1 = -2z_2^2 \ ;$$

this integrates to (fig. 78)

$$\ln|y_1| = -\frac{1}{2z_2} - \frac{1}{2}\ln|z_2| + c \ .$$

In x_1, x_2 variables, this gives us figure 77 (with just the single branch $\mathscr{F}_1$).

Example 3.

$$\dot{x}_1 = 4x_2^2 + x_1 x_2 - 2x_1^2 = x_1(4x_1^{-1}x_2 + x_2 - 2x_1) \ ,$$

$$\dot{x}_2 = x_2^2 + 2x_1 x_2 + 2x_1^3 = x_2(x_2 + 2x_1 + 2x_1^3 x_2^{-1}) \ .$$

The Newton open polygon (fig. 79) consists of three vertices and two edges, $\Gamma_1^{(1)} \supset Q_4, Q_3$ and $\Gamma_2^{(1)} \supset Q_3, Q_2, Q_1$. Here, $R_1 = (-2, 1)$ and $R_2 = (-1, 1)$. The vertices $\Gamma_1^{(0)} = Q_4$ and $\Gamma_3^{(0)} = Q_1$ are of the first type, and the behavior of the integral curves in the sets $\mathscr{U}_1^{(0)}(\varepsilon)$ and $\mathscr{U}_3^{(0)}(\varepsilon)$ is trivial (fig. 80), but the vertex

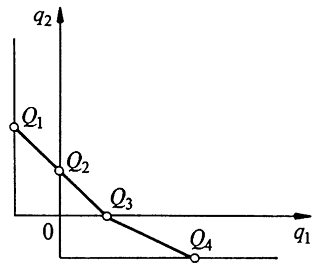

Fig. 79

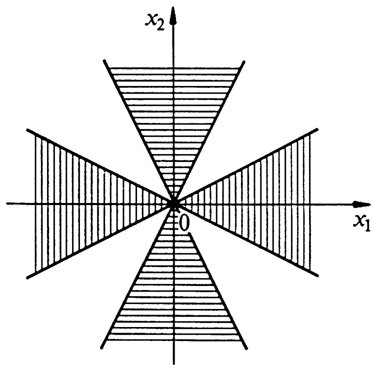

Fig. 80

$\Gamma_2^{(0)} = Q_3$ is of the third type, and its set ${}_1\mathcal{U}_2^{(0)}(\varepsilon)$ contains the sets $\mathcal{U}_1^{(0)}(\varepsilon)$ and $\mathcal{U}_1^{(1)}(\varepsilon)$, which correspond to the vertex $\Gamma_1^{(0)} = Q_4$ and to the edge $\Gamma_1^{(1)}$. For the vertex $\Gamma_2^{(0)}$ we have $\tilde{R}_* = (1,0)$, $R^* = R_2$, and $\Lambda = F_{Q_3} = (-2,2)$. Since the scalar products $\langle \tilde{R}_*, \Lambda \rangle = -2$ and $\langle R^*, \Lambda \rangle = 4$ have opposite signs, the line $\mathbf{L}$ intersects the sector ${}_1\mathbf{V}$ at the interior points. This is case d) of section 2.3; the integral curves in the curvilinear sectors of the set ${}_1\mathcal{U}_2^{(0)}(\varepsilon)$ behave as in figure 69. In order to more accurately define the behavior of the unique integral curve which passes through the origin, we make use of the edge $\Gamma_1^{(1)}$. We need to perform a power transformation and find a fixed point on the vertical axis. Instead, we will seek a solution which expresses x_2 as a power series of x_1 (that is, instead of searching for an analog of the second method of resolving algebraic curves, we will employ an analog of the first method). Since $R_1 = (-2, 1)$, then $\gamma_1 = 2$, and, for the truncated system (corresponding to the edge $\Gamma_1^{(1)}$)

$$\dot{x}_1 = -2x_1^2\,, \qquad \dot{x}_2 = 2x_1x_2 + 2x_1^3$$

we must find a solution of the form $x_2 = bx_1^2$. We write this system as a single equation

$$\frac{dx_2}{dx_1} = -\frac{x_2 + x_1^2}{x_1} \,.$$

Substituting $x_2 = bx_1^2$ into this equation, we obtain an equation for b, $2b = -(b+1)$, whence $b = -1/3$. Consequently, the one integral curve which passes through the origin in the set ${}_1\mathscr{U}_2^{(0)}(\varepsilon)$ is defined by

$$x_2 = -\tfrac{1}{3}x_1^2 + \cdots \,.$$

In order to consider solutions corresponding to the edge $\Gamma_2^{(1)}$, we make the transformation

$$y_1 = x_1 \,, \qquad y_2 = x_1^{-1}x_2 \,,$$

and obtain

$$\dot{y}_1 = y_1(4y_1y_2^2 + y_1y_2 - 2y_1) = y_1^2(4y_2^2 + y_2 - 2) \,,$$

$$\dot{y}_2 = y_2(-4y_1y_2^2 + 4y_1 + 2y_1^2y_2^{-1}) = y_1y_2(-4y_2^2 + 4 + 2y_1y_2^{-1}) \,.$$

This is of type I, section 3.3. For the fixed points $y_1 = 0$, $y_2 = y_2^0$ we obtain the equation $y_2^2 = 1$, hence $y_2^0 = \pm 1$. The eigenvalues at these points are $\lambda_1 = 2 \pm 1$, $\lambda_2 = \pm 6$. The point with $y_2^0 = 1$ is a saddle (one integral curve, $y_2 = 1 + \cdots$, passes through it), while the point with $y_2^0 = -1$ is a node (an infinite number of integral curves, $y_2 = -1 + \cdots$, pass through the point). Consequently, there is one curve $\mathscr{F}_2$ ($x_2 = x_1 + \cdots$) and an infinite number of curves $\mathscr{F}_3$ ($x_2 = -x_1 + \cdots$) (fig. 81).

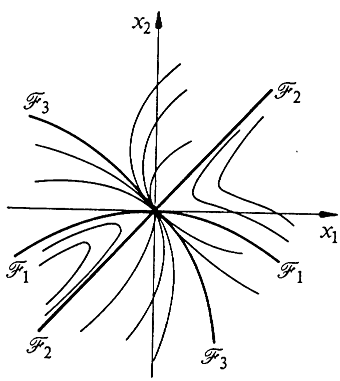

Fig. 81

Exercise 1. Outline the behavior of the integral curves of the following systems near the origin:

1) $\dot{x} = y - x^2 + 4y^2$,
   $\dot{y} = axy + 3y^2 + x^4$, $a = -1, -2, -3, -4$;
2) $\dot{x} = x^3$,
   $\dot{y} = x^4 + x^2 + x^2y + 2xy + y^2$;
3) $\dot{x} = xy$,
   $\dot{y} = x^2 - y^2$;
4) The system of example 1 with $a = 0$.

3.6. Comparison with Frommer's Method

Frommer [1928] suggested a method of investigating integral curves in the neighborhood of a non-elementary singular point. This method, while it is different in terms of technical details, actually has much in common with the local method. Frommer used an analog of the Newton open polygon (section 1.7, Chapter I), reduction of singularities to elementary singular points (similar to the first method in section 2.8 of Chapter I), and the division of the neighborhood of the singular point into curvilinear parts ("Frommer's regions"). These Frommer's regions are generally analogous to the components of the sets $\mathcal{U}_j^{(d)}(\varepsilon)$. But Frommer's method differs from the local method in terms of investigating the integral curves inside these regions: specifically, Frommer did not have the idea of the normal form. Instead of finding the normal form, Frommer proves a series of theorems which establish a connection between the behavior of the right-hand sides of the equations of the system in a Frommer's region (and along its boundary) and the behavior of the integral curves in that region. In those cases when the disposition of the integral curves is determined by the higher-order terms of the normal form, Frommer's method requires an even more detailed investigation of the properties of the right-hand sides. This has produced a considerable body of work (see, for example: Nemytskiĭ and Stepanov, 1947; Kukles, 1958, 1964; Andreev, 1970; Andreev and Gleĭser, 1972).

Frommer's method is preferable to the method of the normal form only in the following situations: if the right-hand sides in the given system are not analytic, but have only a finite number of derivatives, and if the corresponding finite Taylor expansions are insufficient for calculating the determining terms in the normal form.

We note that the local method of investigating the neighborhood of non-elementary singularity consists of a sequence of algebraic operations. This sequence of operations and the corresponding calculations can easily be programmed on a computer (see Berezovskaya and Kreitzer, 1975).

§4. On Distinguishing Between a Center and a Focus

4.1. Introduction

Suppose that for a real analytic system

$$\dot{x}_i = \varphi_i(X) \ , \qquad \varphi_i(0) = 0 \ , \qquad i = 1, 2 \tag{1}$$

there is no set $\mathscr{U}_j^{(d)}(\varepsilon)$ (corresponding to an edge or a vertex of the Newton open polygon $\hat{\Gamma}$) in which an integral curve approaches the origin. Then the integral curves in the neighborhood $\mathscr{U}$ must be either closed curves (the case of a center) or spirals which wind out of the singular point (a focus). This part of the chapter is devoted to a method for distinguishing between these two cases and investigating their properties. Clearly, we can consider instead of the system (1) a single differential equation

$$\frac{dx_2}{dx_1} = \frac{\varphi_2(X)}{\varphi_1(X)} \ . \tag{2}$$

Let us first consider the simplest case, when the open polygon $\hat{\Gamma}$ consists of a single edge with the unit vector $R = (-1, 1)$. Then in the truncated system

$$\dot{x}_i = \hat{\varphi}_i(X) \ , \qquad i = 1, 2 \ , \tag{3}$$

the $\hat{\varphi}_i$ are homogeneous polynomials of odd degree m. We will consider other cases in section 4.7.

But for most of this part, we will assume that:

1) the expansions of the φ_i begin with homogeneous forms $\hat{\varphi}_i(X)$ of odd degree m, and

2) system (1) has no real directions of approach, i.e.,

$$x_1 \hat{\varphi}_2(X) - x_2 \hat{\varphi}_1(X) \neq 0 \text{ for } X \neq 0 \ , \qquad X \in \mathbf{R}^2 \ . \tag{4}$$

4.2. Trigonometric Theory

We introduce polar coordinates r, θ:

$$x_1 = r\cos\theta \ , \qquad x_2 = r\sin\theta \ .$$

Then system (1) becomes:

$$\dot{r} = \varphi_1 \cos\theta + \varphi_2 \sin\theta \ ,$$

$$\dot{\theta} = \frac{1}{r}(-\varphi_1 \sin\theta + \varphi_2 \cos\theta) \ ,$$

which is equivalent to the equation

$$\frac{dr}{d\theta} = \frac{r(\varphi_1 \cos\theta + \varphi_2 \sin\theta)}{-\varphi_1 \sin\theta + \varphi_2 \cos\theta} \ .$$

Condition (4) assures us that the denominator never vanishes for $r \neq 0$. Consequently, this equation can be written in the form

$$dr/d\theta = f(r,\theta) = \sum_{k=1}^{\infty} r^k f_k(\theta) \ , \tag{5}$$

where the f_k are 2π-periodic functions, analytic in some region $|\operatorname{Im}\theta| < \varepsilon$.

We now consider the truncated equation

$$dr/d\theta = r f_1(\theta) \ . \tag{6}$$

Separating variables, we can easily obtain the solution

$$r = c \exp \int f_1(\theta)\, d\theta \ . \tag{7}$$

We can find the mean value a of $f_1(\theta)$:

$$a = \frac{1}{2\pi} \int_0^{2\pi} f_1(\theta)\, d\theta \ ; \tag{8}$$

then solution (7) can be written in the form

$$r = c g(\theta) \exp a\theta \ , \tag{9}$$

where

$$g(\theta) = \exp \int_0^{\theta} (f_1(\theta_1) - a)\, d\theta_1 \ . \tag{10}$$

This is a 2π-periodic function with no real zeros.

If $a \neq 0$, then the solutions (9) are spirals (the origin is a focus); if $a = 0$, the solutions (9) are closed curves (the origin is a center). But all of this was established for the truncated equation (6). In order to examine the behavior of the complete equation (5), we introduce a "generalized polar radius" $\tilde{r} = r g^{-1}(\theta)$. Then equation (5) takes the form

$$d\tilde{r}/d\theta = a\tilde{r} + \sum_{k=2}^{\infty} \tilde{r}^k \tilde{f}_k(\theta) \ , \tag{11}$$

where

$$\tilde{f}_k(\theta) = f_k g^{k-1} , \qquad k = 2, 3, \ldots \tag{12}$$

are analytic, 2π-periodic functions, and the series converges for sufficiently small $\tilde{r}$. We attempt to simplify equation (11) by means of a formal transformation of the generalized polar radius:

$$\tilde{r} = \rho + \sum_{k=2}^{\infty} h_k(\theta)\rho^k \equiv \rho + h , \tag{13}$$

where the $h_k(\theta)$ are analytic, 2π-periodic functions, though the series may diverge for every $\rho \neq 0$.

Theorem 1. *There exists a formal change of the coordinate* (13) *such that equation* (11) *takes the form*

$$d\rho/d\theta = a\rho + \sum_{k=2}^{\infty} b_k \rho^k \equiv a\rho + b(\rho) , \tag{14}$$

where all the b_k vanish if $a \neq 0$.

Proof. Differentiating equation (13) with respect to θ and substituting expressions (11) and (14) for the derivatives yields

$$a\tilde{r} + \sum_{k=2}^{\infty} \tilde{f}_k \tilde{r}^k = \left(1 + \sum_{k=2}^{\infty} k h_k \rho^{k-1}\right)(a\rho + b(\rho)) + \sum_{k=2}^{\infty} \rho^k \frac{dh_k}{d\theta} .$$

Substituting expression (13) for $\tilde{r}$ and collecting terms, we obtain an identity in ρ and θ:

$$ah + \sum_{k=2}^{\infty} \tilde{f}_k(\rho + h)^k = b + a\sum_{k=2}^{\infty} k h_k \rho^k + b\sum_{k=2}^{\infty} k h_k \rho^{k-1} + \sum_{k=2}^{\infty} \rho^k \frac{dh_k}{d\theta} .$$

This identity will be satisfied if the coefficients of ρ^k on each side are equal. This condition yields the equations

$$a h_k + \{\tilde{f}\}_k = b_k + akh_k + \sum_{i+j=k+1} b_i j h_j + dh_k/d\theta , \tag{15}$$

where $\{\tilde{f}\}_k$ denotes the coefficient of ρ^k in the series $\sum \tilde{f}_l(\rho + h)^l$. We let

$$c_k(\theta) = \{\tilde{f}\}_k - \sum_{i+j=k+1} b_i j h_j ; \tag{16}$$

it is evident that each $c_k(\theta)$ is a 2π-periodic function, being a polynomial in h_j, b_j, and f_j with indices $j < k$. We can then write equation (15) in the form

$$dh_k/d\theta = a(1 - k)h_k - b_k + c_k . \tag{17}$$

We must now show that this equation has a periodic solution h_k for some value of the constant b_k. Then we can prove the theorem by induction on k.

If $a = 0$, then

$$h_k = -b_k\theta + \int c_k(\theta)\,d\theta \ , \tag{18}$$

and the function $h_k(\theta)$ will be periodic if we choose for the constant b_k the mean value of the function $c_k(\theta)$:

$$b_k = \frac{1}{2\pi}\int_0^{2\pi} c_k(\theta)\,d\theta \ . \tag{19}$$

If $a \neq 0$, it is more convenient to work with Fourier series. Let

$$c_k = c_k^0 + \sum_{l=1}^{\infty} (c_{kl}^* \cos l\theta + c_{kl}^{**} \sin l\theta) \ ,$$

$$h_k = h_k^0 + \sum_{l=1}^{\infty} (h_{kl}^* \cos l\theta + h_{kl}^{**} \sin l\theta) \ .$$

Equation (17) will be satisfied for $b_k = 0$ if the coefficients of $\cos l\theta$ and $\sin l\theta$ are the same on both sides of the equation. We thus obtain a system of equations for the coefficients h_{kl}:

$$a(1-k)h_k^0 + c_k^0 = 0 \ ;$$

$$\begin{cases} lh_{kl}^{**} = a(1-k)h_{kl}^* + c_{kl}^* \ , \\ -lh_{kl}^* = a(1-k)h_{kl}^{**} + c_{kl}^{**}, \qquad l = 1, 2, \ldots \end{cases}$$

For every l, the last system has a unique solution since its discriminant is equal to $a^2(k-1)^2 + l^2$. The rates of growth of the coefficients h_{kl}^{**} and h_{kl}^* with respect to l are like those of c_{kl}^{**} and c_{kl}^*. The function $h_k(\theta)$ is, therefore, analytic in the same region $|\mathrm{Im}\,\theta| < \varepsilon$ in which the function $c_k(\theta)$ is analytic. □

Example 1. Let us calculate the constants b_2 and b_3 for equation (15) in the case $a = 0$. According to (16), (12), and (19), we have

$$c_2 = \tilde{f}_2 = f_2 g \ ,$$

$$b_2 = \frac{1}{2\pi}\int_0^{2\pi} f_2 g\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} f_2(\theta)\exp\left(\int_0^{\theta} f_1(\theta_1)\,d\theta_1\right)d\theta \ .$$

Similarly, we obtain

$$c_3 = \tilde{f}_3 + \tilde{f}_2 2h_2 - 2b_2h_2 = f_3 g^2 + 2h_2(f_2 g - b_2) \ ,$$

$$b_3 = \frac{1}{2\pi}\int_0^{2\pi} c_3\,d\theta \ ,$$

where, according to (18)

$$h_2(\theta) = \int_0^\theta (f_2 g - b_2)\,d\theta \ .$$

4.3. The Study of Normal Forms

In the last section, we showed that, with the introduction of polar coordinates and their subsequent transformation, equation (2) transforms into the normal form (14). Let us consider the different cases.

1) If $a \neq 0$, the normal form (14) is the equation $\dot{\rho} = a\rho$. This is the case of a structurally stable focus; it remains such a focus under any "perturbation" of system (1) if the order of the perturbation is greater than m. In this case, the normalizing transformation is convergent (see Bruno, 1971, 1972a).

2) $a = 0$, $b(\rho) \not\equiv 0$. In this case the solutions of equation (14) are spirals

$$\theta = \int \frac{d\rho}{b(\rho)} = -\frac{1}{(n-1)b_n\rho^{n-1}}(1 + \cdots) \ ,$$

where b_n is the first non-zero coefficient in the series $b(\rho) = \sum b_k \rho^k$. In theorem 3 of § 2 in Chapter III, we will show that with the transformation

$$\rho = \sum_{k=1}^{\infty} g_k \tilde{\rho}^k$$

equation (14) can be put into polynomial form:

$$d\tilde{\rho}/d\theta = b_n \tilde{\rho}^n + \tilde{b}_{2n-1}\tilde{\rho}^{2n-1} \ .$$

This is the case of a non-structurally stable focus (see sect. 1.11 in Chapter II). As a rule, the normalizing transformation diverges in this case (see Bruno, 1971, 1972a; Osipov, 1973), but there exists a smooth normalizing transformation (Bibikov, 1971, 1973; Tokarev, 1977).

3) If $a = 0$ and $b(\rho) \equiv 0$, the solutions of equation (14) are concentric circles $\rho = \text{const}$. That is, the singular point is a center. The normalizing transformation converges.

Note that the transformation from cartesian coordinates x_1, x_2 to polar coordinates r, θ blows up the singular point $X = 0$ into the invariant circle $r = 0$. After such blowing up, we applied the method of studying solutions near an invariant closed solution without singular points. The specific character of the original system (1) was thereby lost.

4.4. On the Computability of the Coefficients of the Normal Form

This aspect is manifested, in particular, by the fact that the functions $f_k(\theta)$ in equation (5) are rational functions of $\sin\theta$ and $\cos\theta$; they are, in fact, homo-

geneous functions of degree $k-1$ of these arguments. Consequently, and in agreement with (8), the value of a will be an algebraic function of the coefficients of the polynomials $\hat{\varphi}_i$ (see Khinchin, 1955, § 65). But the values of the b_k (if $a = 0$) will in this case be transcendental functions of those coefficients. For example, consider the system

$$\dot{x}_i = \hat{\varphi}_i(X) + \psi_i(X) \ , \qquad i = 1, 2 \ , \tag{20}$$

where the $\hat{\varphi}_i$ and ψ_i are homogeneous polynomials of degree m and $m+2$, respectively, and $a = 0$. Hence, $f_2 \equiv 0$ in expansion (5). Then, according to the formulas of example 1, we have

$$\begin{gathered} h_2 = c_2 = 0 \ , \qquad b_2 = 0 \ , \\ b_3 = \frac{1}{2\pi}\int_0^{2\pi} f_3 g^2 \, d\theta = \frac{1}{2\pi}\int_0^{2\pi} f_3(\theta)\exp\left(2\int_0^{\theta} f_1(\theta_1)\,d\theta_1\right)d\theta \ . \end{gathered} \tag{21}$$

Here the coefficients of the polynomials $\hat{\varphi}_i$ are included in f_1 and appear in the exponents of e; that is, they do not enter into formula (21) in an algebraic manner. Specifically, the conditions for a center ($b_k = 0$, $k = 2, 3, \ldots$) are not algebraic conditions on the coefficients of the original system (1). Recall that for an elementary singular point the coefficients of the normal form are polynomials in the coefficients of the original system, and that the conditions for a center are, in that case, algebraic conditions on those coefficients.

In this connection, we note the assertion of Arnol'd [1970] that the case of a center is always distinguished by algebraic conditions on the coefficients of the original system (1). In light of the above, we see that this assertion is correct for elementary singular points but is not true for non-elementary ones. Il'yashenko [1972] gave a rigorous proof of the non-algebraic nature of the conditions for a center. He considered a system like (20) where $m = 3$ and the coefficients of the forms $\hat{\varphi}_1$ and $\hat{\varphi}_2$ formed a five-parameter family.

The conditions of a center are non-algebraic only for coefficients of the forms $\hat{\varphi}_1$ and $\hat{\varphi}_2$. If these coefficients are fixed, then the conditions for a center are algebraic for the remaining coefficients of system (1), since they do not appear in the exponent in formula (21) (see Il'yashenko, 1972). Apparently, this property also holds in the case when the coefficients of the forms $\hat{\varphi}_i$ are not fixed but belong to a family with a small number of parameters. It would be interesting to find an exact bound on the number of parameters under which the conditions for a center remain algebraic. Il'yashenko's proof tells us only that this bound is no greater than four. Perhaps it is even less.

4.5. Integration of the Truncated System (3) and Computation of the Constant a

If we take $u = x_1^{-1}x_2$, then system (3) is equivalent to the system

$$(\ln x_1)^{\bullet} = x_1^{m-1}\hat{\varphi}_1(1, u) \ ,$$

$$(\ln u)^{\cdot} = x_1^{m-1}[\hat{\varphi}_2(1,u)u^{-1} - \hat{\varphi}_1(1,u)] .$$

This system corresponds to the equation

$$\frac{d\ln x_1}{du} = \frac{\hat{\varphi}_1(1,u)}{\hat{\varphi}_2(1,u) - \hat{\varphi}_1(1,u)u} \equiv \delta(u) . \tag{22}$$

Thanks to condition (4), the denominator of the right-hand side never vanishes for real u, and it is a polynomial of degree $m + 1$. The degree of the numerator never exceeds m. If we express the right-hand side of the equation (22) as a sum of simple fractions we obtain

$$\delta(u) = \sum_k \sum_{l=1}^{l_k} \left(\frac{\alpha_{kl}}{(u-\beta_k)^l} + \frac{\bar{\alpha}_{kl}}{(u-\bar{\beta}_k)^l} \right) , \tag{23}$$

where β_k and $\bar{\beta}_k$ are the roots of the equation

$$\hat{\varphi}_2(1,u) - u\hat{\varphi}_1(1,u) = 0$$

of multiplicity l_k, such that $2\sum l_k = m + 1$. We will assume that $\operatorname{Im}\beta_k > 0$. From (22), we have

$$\ln x_1 = \int \delta(u)\,du ,$$

where the integral is taken along the real axis. Using expression (23), it is easy to express this integral as a sum of elementary functions: logarithms, arctangents, and rational fractions (see Khinchin, 1955, §§ 60, 61). Because of its complexity, we will not write out the integral in the general case here. We will only calculate the change in the value of x_1 along an integral curve in a single circuit about the origin, $X = 0$. One circuit around this point corresponds to two integrations over the real axis in the complex plane of u. Consequently,

$$\ln x_1 - \ln x_1^0 = 2\int_\gamma \delta(u)\,du = 2\pi a ,$$

where the integral represents a Cauchy integral and γ is a closed contour corresponding to the real axis. According to the theory of analytic functions, the evaluation of such an integral reduces to finding the residues of the function $\delta(u)$ (see Privalov, 1960, Ch. 7). In agreement with (23)

$$2\int_\gamma \delta(u)\,du = 2\pi i \sum_k (\alpha_{k1} - \bar{\alpha}_{k1}) = -4\pi \sum_k \operatorname{Im}\alpha_{k1} .$$

Therefore

$$a = -2\sum_k \operatorname{Im}\alpha_{k1} . \tag{24}$$

Example 2. Consider the homogeneous system

$$\begin{aligned} \dot{x}_1 &= -x_1x_2^2 - x_2^3 \equiv \varphi_1 , \\ \dot{x}_2 &= \omega^2 x_1^3 + (1+\omega^2)x_1x_2^2 - x_2^3 \equiv \varphi_2 . \end{aligned} \tag{25}$$

Condition (4) is satisfied since the polynomial

$$x_1\varphi_2 - x_2\varphi_1 = \omega^2 x_1^4 + (1+\omega^2)x_1^2x_2^2 + x_2^4$$

has no real roots. Equation (22) is

$$\frac{d\ln x_1}{du} = \frac{-u^2-u^3}{u^4+(1+\omega^2)u^2+\omega^2} = \delta(u)\ . \tag{26}$$

The roots of the denominator are $\beta_1 = i = \bar{\beta}_3, \beta_2 = i\omega = \bar{\beta}_4$. If $\omega \neq \pm 1$, all roots are simple. Let us assume that

$$\omega > 0, \omega \neq 1\ ; \tag{27}$$

then expression (23) is

$$\delta(u) = \frac{\alpha_1}{u-i} + \frac{\bar{\alpha}_1}{u+i} + \frac{\alpha_2}{u-i\omega} + \frac{\bar{\alpha}_2}{u+i\omega}\ .$$

By writing this as a single fraction, with the same denominator as that in expression (26), and then equating the coefficients of like terms in the numerators of the two fractions, we can obtain a system of linear equations for $\alpha_1, \bar{\alpha}_1, \alpha_2$ and $\bar{\alpha}_2$. Solving this system yields

$$\alpha_1 = \frac{1-i}{\omega^2-1}\ , \qquad \alpha_2 = -\frac{\omega^2}{\omega^2-1}\left(1-\frac{i}{\omega}\right)\ .$$

Then the integral of equation (26) is

$$\ln x_1 = \operatorname{Re}\alpha_1 \ln(1+u^2) - \operatorname{Im}\alpha_1 \operatorname{arctg} u$$

$$+ \operatorname{Re}\alpha_2 \ln(1+\omega^{-2}u^2) - \operatorname{Im}\alpha_2 \operatorname{arctg}\frac{u}{\omega} + c\ .$$

In agreement with (24), we have

$$a = -2(\operatorname{Im}\alpha_1 + \operatorname{Im}\alpha_2) = -2\left(-\frac{1}{\omega^2-1} + \frac{\omega}{\omega^2-1}\right)$$

$$= -2\frac{\omega-1}{\omega^2-1} = -\frac{2}{\omega+1}\ .$$

Consequently, by condition (27), the integral curves of system (25) form a focus. Theorem 1 informs us that it will remain a focus, even if terms of order greater than 3 are inserted into the right-hand sides of expressions (25).

Exercise 1. Calculate the value of a for system (25) if $\omega = 1$ and evaluate the integral explicitly.

4.6. Normalization in Complex Coordinates

We introduce complex coordinates

$$z_1 = x_1 + ix_2 , \qquad z_2 = x_1 - ix_2 , \tag{28}$$

i.e.,

$$x_1 = \frac{1}{2}(z_1 + z_2) , \qquad x_2 = \frac{1}{2i}(z_1 - z_2) .$$

Thus, the complex-conjugate coordinates $z_1 = \bar{z}_2$ correspond to the real coordinates x_1, x_2. System (1) takes the form

$$\begin{aligned} \dot{z}_1 &= \varphi_1 + i\varphi_2 = \varphi(z_1, z_2) , \\ \dot{z}_2 &= \varphi_1 - i\varphi_2 = \bar{\varphi}(z_2, z_1) . \end{aligned} \tag{29}$$

We make the power transformation

$$v = z_2 z_1^{-1} . \tag{30}$$

and system (29) becomes the system

$$\begin{aligned} \dot{z}_1 &= \varphi(z_1, z_1 v) , \\ \dot{v} &= v(z_2^{-1}\bar{\varphi}(z_2, z_1) - \varphi(z_1, z_2) z_1^{-1}) , \end{aligned}$$

which corresponds to the equation

$$\frac{d \ln z_1}{d \ln v} = \frac{v\varphi}{\bar{\varphi}(z_1 v, z_1) - v\varphi(z_1, z_1 v)} = \sum_{k=0}^{\infty} \xi_k(v) z_1^k , \tag{31}$$

where the $\xi_k = \sum_{l=-\infty}^{\infty} \xi_{kl} v^l$ are Laurent series, and the exponents k, l are integers. This equation can be put into normal form with a coordinate change of the form

$$z_1 = w \sum_{k=0}^{\infty} \zeta_k(v) w^k = w \sum_{k=0}^{\infty} w^k \sum_{l=-\infty}^{+\infty} \zeta_{kl} v^l . \tag{32}$$

Under transformations (28) and (30), the singular point $X = 0$ is blown up into the unit circle $|v| = 1, z_1 = 0$, and a circuit about the point $X = 0$ in the $\mathbf{R}_0^2$ plane corresponds to a double traverse of this circle. The Laurent series $\xi_k(v)$ and $\zeta_k(v)$ converge in some ring $1 - \varepsilon < |v| < 1 + \varepsilon$.

The complex coordinates can be associated with polar coordinates in the following manner: $z_1 = re^{i\theta}$, $v = e^{-2i\theta}$. The Laurent series in v are just Fourier series in 2θ. For the truncated equation we therefore have

$$\frac{d \ln z_1}{d \ln v} = \frac{d \ln r + i d\theta}{-2i d\theta} = i \frac{d \ln r}{2 d\theta} - \frac{1}{2} = -\frac{1}{2} + \frac{a}{2} i + \tilde{g}(v) ,$$

where $\tilde{g}(v)$ is a Laurent series with no constant term. In the standard manner, we can show that under the transformation (32), equation (31) is carried into the normal form

$$\frac{d\ln w}{d\ln v} = \sum_{k=0}^{\infty} \sum_{l=-\infty}^{+\infty} \beta_{kl} v^l w^k ,$$

consisting solely of resonant terms, for which

$$k\lambda + l = 0 , \tag{33}$$

where $\lambda = -\frac{1}{2} + \frac{a}{2} i$ (see Bruno, 1971, 1972a, § 11; 1970b).

If $a \neq 0$, then the only real solution of equation (33) is the trivial one $k = l = 0$, and the normal form is linear:

$$\frac{d\ln w}{d\ln v} = \lambda .$$

If $a = 0$, then integral solutions of equation (33) have the form $k = 2l$, and the normal form is

$$\frac{d\ln w}{d\ln v} = -\frac{1}{2} + \sum_{l=1}^{\infty} \gamma_l v^l w^{2l} \tag{34}$$

or (for $\rho = w\sqrt{v}$)

$$\frac{d\ln \rho}{d\ln v} = \sum_{l=1}^{\infty} \gamma_l \rho^{2l} . \tag{34'}$$

The reality relations for coordinates v, z_1 are $\bar{v} = v^{-1}, \bar{z}_1 = vz_1$. So the reality conditions for equation (31) are

$$\xi_{00} + \bar{\xi}_{00} = -1 , \qquad \xi_{k,l} + \bar{\xi}_{k,k-l} = 0 , \qquad k + |l| \neq 0 .$$

As in theorem 3 of section 1.9, we can prove the existence of the normalizing transformation (32) with $\zeta_{k,l} = \bar{\zeta}_{k,k-l}$, which preserves both the reality relations and the reality conditions. That is, in the normal form (34) all the coefficients γ_l are pure imaginary. Since $\ln v = -2i\theta$, then the normal form (34′) must coincide with the normal form (14). Hence, in the normal form (14)

$$b_k = \begin{cases} 0, & k \text{ even} , \\ -2i\gamma_{(k-1)/2} , & k \text{ odd} . \end{cases}$$

That is, the right-hand part of equation (14) contains only odd powers of ρ.

4.7. Classification

We will confine ourselves here to the following classification of the cases in which the problem of distinguishing between a center and a focus arises for

system (1):

1. The Newton open polygon $\hat{\Gamma}$ consists of a single edge $\Gamma_1^{(1)}$ with unit vector $R = (r_1, r_2)$, where both r_1 and r_2 are odd integers.

2. The open polygon $\hat{\Gamma}$ contains only one edge $\Gamma_1^{(1)}$, the unit vector R of which has an even component.

3. The open polygon $\hat{\Gamma}$ has more than one edge.

We consider these cases in order below.

1. After the introduction of new coordinates y_1, y_2 according to the formula

$$x_1 = y_1^n , \qquad x_2 = y_2^m ,$$

where $m = |r_1|$ and $n = |r_2|$, and the change of time variable $dt = y_1^{n-1} y_2^{m-1} \, d\tau$, system (1) becomes

$$dy_1/d\tau = y_2^{m-1} \varphi_1(y_1^n, y_2^m) ,$$

$$dy_2/d\tau = y_1^{n-1} \varphi_2(y_1^n, y_2^m) .$$

The right-hand sides of the truncated form of this system are homogeneous polynomials, and condition (4) is satisfied for this system. The analysis of such systems is given in previous sections. Lyapunov [1935b, sects. 9–20] examined some subcases with $R = (n, -1)$, introducing new special functions.

2. In this case, the truncated system always has a center since its integral curves are symmetric with respect to one of the coordinate axes. Indeed, if r_1 is even, the truncated system transforms into itself under a change of the sign of x_1. This case reduces to the case examined in sections 4.1–4.3 under the coordinate change

$$x_1 = u^n , \qquad x_2 = \frac{(v + \sqrt{u^2 + v^2})^{2m} - u^{2m}}{2(v + \sqrt{u^2 + v^2})^m} ,$$

where $m = |r_1|$ and $n = |r_2|$. Switching to polar coordinate r, θ ($u = r\cos\theta$, $v = r\sin\theta$), we obtain an equation (5) which has all the necessary properties. By theorem 1, it can be transformed into a normal form with $a = 0$.

Lyapunov [1935b, sects. 9–20] again considered some subcases with $R = (n, -1)$ using special functions. In particular, in section 19 he calculated the value of b_2 in the normal form. It turns out that b_2 depends algebraically on the coefficients of the right-hand sides. Apparently, this is a consequence of the fact that the coefficients of the truncation depend only on one parameter.

3. At this time, no example of this case has been studied. The general approach of the local method, consisting of solving system (1) separately in each set $\mathcal{U}_j^{(d)}(\varepsilon)$ and the subsequent "sewing together", can give an answer here. But questions of sewing the parts of the solutions from the different sets $\mathcal{U}_j^{(d)}(\varepsilon)$ have not been sufficiently worked out (see section 3.4). This case to some degree resembles the problem of investigating the neighborhood of a closed integral curve containing one or several singular points. The investigation of this situation is an important

problem of the theory of differential equations. Similar problems, with a large number of variables, arise in celestial mechanics. As an example, we can mention the determination of periodic solutions of the second kind in the restricted three-body problem.

Remark. There is a large literature in which conditions for a center are derived for these and other particular cases (see, for example, Sadovsky, 1976, 1978).

Exercise 2. Explain whether a focus or a center results when we take $|a| < 2$ in example 2 of section 3.5.

4.8. The Neighborhood of a Periodic Solution

Let the system

$$\dot{x}_i = \varphi_i(x_1, x_2) \ , \qquad i = 1, 2 \tag{35}$$

be analytic in some region $\mathscr{G}$, and suppose there is a periodic solution $\mathscr{M}$ of period τ:

$$x_i = \chi_i(t) = \chi_i(t + \tau) \ , \qquad i = 1, 2 \ ,$$

where the functions χ_i are analytic. In a tubular neighborhood of the periodic solution $\mathscr{M}$ we introduce new coordinates r, t by the formulas

$$\begin{aligned} x_1 &= \chi_1(t) - r\varphi_2(\chi_1(t), \chi_2(t)) \ , \\ x_2 &= \chi_2(t) + r\varphi_1(\chi_1(t), \chi_2(t)) \ . \end{aligned} \tag{36}$$

Here r measures the distance from the point x_1, x_2 to the periodic solution $\mathscr{M}$ along a normal; if $r = 0$, then x_1, x_2 is on $\mathscr{M}$. Transformation (36) is one-to-one in the tubular neighborhood $|r| < \varepsilon$ for some $\varepsilon > 0$. In fact, the Jacobian of transformation (36) is given by

$$\Delta = -\varphi_1^2 - \varphi_2^2 - r(\dot{\varphi}_1\varphi_2 - \dot{\varphi}_2\varphi_1) \ ,$$

where φ_i and $\dot{\varphi}_i$ are calculated on $\mathscr{M}$. As long as there are no stationary points, then $\varphi_1^2 + \varphi_2^2 \neq 0$ and the function $|\dot{\varphi}_1\varphi_2 - \dot{\varphi}_2\varphi_1|$ is bounded from above. Thus, $\Delta \neq 0$ for

$$|r| < \min(\varphi_1^2 + \varphi_2^2)/\max|\dot{\varphi}_1\varphi_2 - \dot{\varphi}_2\varphi_1| \ .$$

After inserting (36) in system (35) we obtain an equation for r:

$$\dot{r} = r\varphi(r, t) = r\varphi(r, t + \tau) \ . \tag{37}$$

After the substitution $\theta = 2\pi t/\tau$, equation (37) takes the form (5). All of the results of sections 4.2 and 4.3 apply; in particular, theorem 1 applies. If $a \neq 0$ in the normal form (14), the periodic solution is a structurally stable limit cycle which is stable if $a < 0$ and unstable if $a > 0$. If $a = 0$, but $b(\rho) \not\equiv 0$, then the periodic

solution $\mathscr{M}$ is a non-structurally stable limit cycle. Its stability is determined by an index l and by the sign of the first non-vanishing coefficient b_l of the series $b(\rho) = b_l\rho^l + \cdots$. If l is even, the limit cycle $\mathscr{M}$ is unstable; if l is odd and $b_l > 0$, it is again unstable; if, however, l is odd and $b_l < 0$, then it is stable. In this setting the limit cycle is said to have multiplicity $l - 1$ since no more than $l - 1$ limit cycles can be generated by small changes in the equations. Finally, if $a = 0$ and $b(\rho) \equiv 0$, then there are other periodic solutions near $\mathscr{M}$. All of these are the level curves of some analytic integral.

These results follow from the arguments in section 4.3. We note that the coordinates r, t or r, θ are "local coordinates for the periodic solution $\mathscr{M}$". Local coordinates are introduced for the general case in section 4.1 of Chapter III.

Chapter III
The Normal Form of a System of n Differential Equations

§ 1. The Normalizing Transformation

1.1. Statement of the Problem

In this chapter, we will examine an autonomous system of ordinary differential equations in the neighborhood of a stationary point. By means of a parallel translation, this point can always be taken as the origin of coordinates. Thus, our system is

$$\dot{X} = AX + \Phi(X) \ , \tag{1}$$

where $X = (x_1, x_2, \ldots, x_n)$ is an n-dimensional vector variable, A is a square matrix of order n, and Φ is a vector function analytic at the origin $X = 0$, the Taylor series of which contains neither constant nor linear terms. Generally speaking, the original coordinates X are not the most convenient for investigating the properties of system (1) near the fixed point $X = 0$. The question then arises: is it possible to simplify system (1) through an invertible change of coordinates which maps the stationary point onto itself? That is, what is the simplest system

$$\dot{Y} = CY + \Psi(Y) \tag{2}$$

to which system (1) can be transformed by a change of coordinates

$$X = BY + \Xi(Y) \ , \tag{3}$$

which is invertible at the stationary point $X = 0$ and maps it onto the point $Y = 0$? In this context, we can take transformation (3) in one of several classes: formal, analytic, smooth, etc. We will consider here only formal and analytic coordinate changes (3), where B is a non-singular matrix and Ξ consists of power series with no constant or linear terms. Let us clarify this. Every function $f(X)$ which is analytic at $X = 0$ can be expressed as a Taylor series

$$f(X) = \sum f_Q X^Q \ , \tag{3'}$$

where $Q = (q_1, q_2, \ldots, q_n)$ and $X^Q = x_1^{q_1} \ldots x_n^{q_n}$. This series converges absolutely in some neighborhood $|x_i| < \rho$ $(i = 1, \ldots, n)$, and, in this neighborhood,

the value of the sum coincides with the functional value $f(X)$. The coefficients of this series satisfy an upper estimate

$$|f_Q| < c\rho^{-(q_1+\cdots+q_n)}, \tag{4}$$

where c is some constant. If the coefficients of the series (3′) do not satisfy such an estimate (4) for some positive c and ρ, then the series diverges in every neighborhood of $X = 0$, and, generally speaking, it is not clear to what function the series can be said to correspond. Power series, for which we make no assumptions about their convergence, are called *formal series*. We often turn to formal series for help in finding the analytic solutions of these and other problems. We begin by finding a formal solution in the form of a power series, and then determine the region in which this series converges and, hence, defines an analytic function. Finding solutions in the class of formal series can be considered to be an independent problem. Even in cases when these series diverge, they may be the Taylor series of infinitely differentiable functions. Arithmetic operations, differentiation, and the substitution of one series in another are all defined for formal series, just as for convergent series.

Thus, in § 1 of the chapter we will consider the problem of simplifying system (1) with the help of formal changes of coordinates (3). Later (in § 3), we will investigate the convergence of these transformations.

Before we try to simplify the non-linear parts of system (1), let us simplify the linear part with a linear change of coordinates:

$$X = BY. \tag{5}$$

1.2. Linear Systems

First, let us briefly recall what we know about Jordan matrices. A Jordan block is a square matrix of the form

$$\begin{pmatrix} \lambda_j & 0 & 0 & \dots & 0 & 0 \\ \sigma & \lambda_j & 0 & \dots & 0 & 0 \\ 0 & \sigma & \lambda_j & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \sigma & \lambda_j \end{pmatrix},$$

where each element along the principal diagonal is the same number λ_j and the elements immediately below the diagonal are all equal to $\sigma \neq 0$; all other elements are zeros. A Jordan matrix is a block-diagonal square matrix

$$J = \{J_1, \dots, J_s\},$$

in which the Jordan blocks $J_1, \dots, J_s$ lie along the principal diagonal and all

elements outside of these blocks vanish. The number σ is the same in all the blocks.

As we know (see Gantmacher, 1959), any matrix A is similar to some Jordan matrix J; that is, there exists a non-singular matrix B such that $J = B^{-1}AB$. The matrix J is called the Jordan normal form of A; the numbers along the main diagonal of T are the eigenvalues of A, $\lambda_1, \ldots, \lambda_n$, and σ is an arbitrary non-zero number.

Every Jordan matrix $C = (c_{ij})$ has the following two properties:

a) it is lower-triangular; i.e., $c_{ij} = 0$ if $i < j$;

b) $c_{ij} = 0$ if $\lambda_i \neq \lambda_j$, where $\lambda_i = c_{ii}$ are the elements of the main diagonal.

Hence, every matrix A is similar to a matrix C with these properties.

Note that transformation (5) takes system (1) into system (2), for which the matrix C is similar to A:

$$C = B^{-1}AB \ .$$

It is therefore always possible to transform system (1) into system (2), where the matrix C possesses properties a) and b). In our further discussions, we will assume that we have already performed a linear transformation such that the matrix A in system (1) has these properties.

1.3. The Existence of the Normalizing Transformation

We will now investigate system (1) by using formal nonlinear changes of coordinates (3), where B is the identity matrix. Such a coordinate change will transform system (1) into system (2), where $C \neq A$. By $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ let us denote the vector of the elements of the main diagonal of A. We set

$$\psi_j = y_j g_j(Y) \equiv y_j \sum_{Q \in \mathbf{N}_j} g_{jQ} Y^Q \ , \qquad j = 1, \ldots, n \ , \tag{6}$$

where $Q = (q_1, q_2, \ldots, q_n)$ and $Y^Q = y_1^{q_1} \ldots y_n^{q_n}$. Since $y_j g_j$ are power series with no constant or linear terms, then

$$\mathbf{N}_j = \{\text{integral } Q\colon q_j \geqslant -1, q_k \geqslant 0\ (k \neq j),\ \|Q\| \geqslant 0\} \ ,$$

where $\|Q\| = q_1 + q_2 + \cdots + q_n$. We will write $\mathbf{N} = \mathbf{N}_1 \cup \cdots \cup \mathbf{N}_n$.

Formal system (2) is called a *normal form* if

1) the matrix C possesses properties a) and b); and

2) in the notation of (6), all the $g_{iQ} = 0$ for $\langle Q, \Lambda \rangle \neq 0$, that is

$$\psi_j = y_j \sum_{\langle Q, \Lambda \rangle = 0} g_{jQ} Y^Q \ , \qquad j = 1, \ldots, n \ .$$

Let us call those terms $y_j g_{jQ} Y^Q$ in expression (6) for which $\langle Q, \Lambda \rangle = 0$ the *resonant terms*. Thus, a normal form is a system (2), where the nonlinear part of the right hand side consists solely of resonant terms.

For the remainder of this chapter, we will employ the following notation:

$$\varphi_i(X) = x_i f_i = x_i \sum f_{iQ} X^Q \ ,$$

$$\psi_i(Y) = y_i g_i = y_i \sum g_{iQ} Y^Q \ ,$$

$$\xi_i(Y) = y_i h_i = y_i \sum h_{iQ} Y^Q \ ,$$

$$\eta_i(Z) = z_i d_i = z_i \sum d_{iQ} Z^Q \ .$$

Symbols from one side of those equations will be carried over to the other side, e.g., $\tilde{\psi}_i = y_i \tilde{g}_i$, etc. Also we will write

$$F_Q = (f_{1Q}, \ldots, f_{nQ}) \ , \qquad F = (f_1, \ldots, f_n) \equiv \sum_Q F_Q X^Q \ ,$$

and we define G and H similarly. We will also use φ_i for the original system (unless noted to the contrary), ψ_i for the normal form, and ξ_i for the normalizing transformation.

Theorem 1. *There exists a formal transformation*

$$x_i = y_i + \xi_i(Y) \ , \qquad i = 1, \ldots, n \tag{7}$$

under which the formal system

$$\dot{x}_i = \lambda_i x_i + \sum_{i>j} a_{ij} x_j + \varphi_i(X) \ , \qquad i = 1, \ldots, n \tag{8}$$

becomes a formal system

$$\dot{y}_i = \lambda_i y_i + \sum_{i>j} a_{ij} y_j + \psi_i(Y) \ , \qquad i = 1, \ldots, n \ , \tag{9}$$

such that $g_{iQ} = 0$ if $\langle Q, \Lambda \rangle \neq 0$. Here the coefficients h_{iQ} for $\langle Q, \Lambda \rangle = 0$ may be given arbitrarily; then the remaining h_{iQ} and the g_{iQ} are uniquely defined. The ξ_i, φ_i, and ψ_i are power series with no terms of less than second order.

Proof. Transformation (7) carries (8) into (9) if the following equations for formal series in Y are satisfied:

$$\sum_{j=1}^{n} \frac{\partial(y_i + \xi_i)}{\partial y_j} \left(\lambda_j y_j + \sum_{j>k} a_{jk} y_k + \psi_j \right) = \lambda_i y_i + \lambda_i \xi_i + \sum_{i>j} a_{ij} y_j + \sum_{i>j} a_{ij} \xi_j + \varphi_i(Y + \Xi) \ , \qquad i = 1, \ldots, n \ .$$

These equations are obtained from system (8) by expressing X in terms of Y according to formula (7) and substituting for $\dot{y}_j$ according to formula (9). Collecting and rearranging terms, we obtain

$$\begin{aligned}
y_i g_i + y_i \sum_{j=1}^{n} \frac{\partial h_i}{\partial y_j} \lambda_j y_j = &-h_i \sum_{i>j} a_{ij} y_j - h_i y_i g_i - y_i \sum_{j=1}^{n} \frac{\partial h_i}{\partial y_j} \sum_{j>k} a_{jk} y_k \\
&- y_i \sum_{j=1}^{n} \frac{\partial h_i}{\partial y_j} y_j g_j + \sum_{i>j} a_{ij} y_j h_j \qquad (10) \\
&+ \varphi_i(y_1 + y_1 h_1, \dots, y_n + y_n h_n) \ , \qquad i = 1, \dots, n \ .
\end{aligned}$$

Writing out the coefficients of $y_i Y^Q$ in the i-th equation (10), we get

$$\begin{aligned}
&g_{iQ} + h_{iQ}\langle Q, \Lambda \rangle \\
&= -\sum_{i>j} h_{iQ-E_j+E_i} a_{ij} - \sum_{P+R=Q} h_{iP} g_{iR} \\
&\quad - \sum_{j=1}^{n} \sum_{j>k} h_{iQ-E_k+E_j}(q_j + 1) a_{jk} - \sum_{P+R=Q} h_{iP} \sum_{j=1}^{n} p_j g_{jR} \qquad (11) \\
&\quad + \sum_{i>j} a_{ij} h_{jQ-E_j+E_i} + \{\varphi_i\}_Q \ , \qquad Q \in \mathbf{N}_i \ , \qquad i = 1, \dots, n \ .
\end{aligned}$$

Here (and later), E_j denotes the j-th unit vector, and $\{\varphi_i\}_Q$ is the coefficient of $y_i Y^Q$ in the series $\varphi_i(Y + \Xi)$. The system of equations (11) is equivalent to system (10), since, when Q runs through the set $\mathbf{N}_i$, the product $y_i Y^Q$ runs through all products of non-negative powers of $y_1, \dots, y_n$.

The set of n-dimensional real vectors can be totally ordered accordingly: a vector P precedes a vector Q if the first non-zero difference from the following set: $\|Q\| - \|P\|, q_1 - p_1, q_2 - p_2, \dots, q_{n-1} - p_{n-1}$, is positive (recall that $\|Q\| = q_1 + \cdots + q_n$). Clearly, only a finite number of the vectors in $\mathbf{N}$ can precede any $Q \in \mathbf{N}$. It is easy to see that, in the right-hand side of (11), only those h_{iP} and g_{jR} appear for which vectors P and R precede the vector Q. This is correct for the 1st, 3rd, and 5th summations in the right-hand side of (11), since the vector $Q - E_j + E_i$ precedes the vector Q if $j < i$; it is true for the second and fourth summations because only those P and R are used for which $\|P\| + \|R\| = \|Q\|$ and $\|P\|, \|R\| > 0$; hence $\|P\|, \|R\| < \|Q\|$. Finally, $\{\varphi_i\}_Q$ contains only those h_{jP} that $\|P\| < \|Q\|$, since the series $\varphi_i(X)$ have no linear terms.

Equations (11) are satisfied when $\|Q\| = 0$, since equations (10) have no linear terms. For $\|Q\| > 0$, equations (11) will be satisfied if we choose

$$\begin{aligned}
&g_{iQ} = 0 \ , \qquad h_{iQ} = \langle Q, \Lambda \rangle^{-1} c_{iQ} \text{ for } \langle Q, \Lambda \rangle \neq 0 \ ; \\
&g_{iQ} = c_{iQ} \ , \qquad h_{iQ} - \text{arbitrary for } \langle Q, \Lambda \rangle = 0 \ ; \qquad Q \in \mathbf{N}_i \ , \qquad i = 1, \dots, n \ .
\end{aligned}$$

Here, c_{iQ} denote the right-hand sides of equations (11). Thus, the g_{iQ} and h_{iQ} ($i = 1, \dots, n$) are defined in the above ordering of Q in correspondence with the assertion of the theorem. □

A normal form can be found even for non-autonomous systems. If the time dependence is periodic or quasi-periodic, then the transformation of the system to normal form corresponds to the transformation of an autonomous system to normal form in the neighborhood of a periodic solution or a family of conditionally periodic solutions (see §§ 4 of chapters II and III and Bruno, 1971, 1972a, § 11). Kostin [1973, 1975] and Sibuya [1958] have treated the normalization of general non-autonomous systems.

1.4. Transformations of the Normal Form

As theorem 1 suggests, the normal form is not uniquely determined by the original system. Suppose that some system (8) is carried by one transformation to normal form (9) and by another transformation to the normal form

$$\dot{z}_i = \lambda_i z_i + \sum_{i>j} \tilde{a}_{ij} z_j + \tilde{\psi}_i(Z) , \qquad i = 1, \dots, n . \tag{12}$$

Then the superposition of one transformation with the inverse of the other will take normal form (9) into normal form (12). Therefore, if we study all transformations of this last type, we will also know all the transformations of the original system to normal forms.

Theorem 2. *If, under the transformation*

$$y_i = \eta_i(Z) , \qquad i = 1, \dots, n \tag{13}$$

normal form (9) *is transformed into normal form* (12), *then* $d_{iQ} = 0$ *for* $\langle Q, \Lambda \rangle \neq 0$.

Proof. The proof is similar to the proof of the preceding theorem. In particular, we obtain this system of equations connecting g, $\tilde{g}$, and d:

$$\begin{aligned} d_{iQ}\langle Q, \Lambda \rangle = & - \sum_{i>j} d_{iQ-E_j+E_i} \tilde{a}_{ij} - \sum_{P+R=Q} d_{iP} \tilde{g}_{iR} \\ & - \sum_{j=1}^{n} \sum_{j>k} d_{iQ-E_k+E_j} (q_j + 1) a_{jk} \\ & - \sum_{P+R=Q} h_{iP} \sum_{j=1}^{n} p_j \tilde{g}_{jR} + \sum_{i>j} a_{ij} h_{jQ-E_j+E_i} \\ & + \{\psi_i\}_Q , \qquad Q \in \mathbf{N}_i , \qquad i = 1, \dots, n . \end{aligned} \tag{14}$$

Just as in the preceding proof, we introduce a total ordering of the vector indices, and we note that, in the right-hand side of (14), only those d_{iP} appear for which P precedes Q. By induction on the vector indices, we shall show that

$$d_{iP} = 0 , \qquad \text{if } \langle P, \Lambda \rangle \neq 0 . \tag{15}$$

The smallest $P \in \mathbf{N}$, according to the ordering used here, is $P = E_n - E_1$. This vector appears only in $\mathbf{N}_1$ and does not appear in the remaining $\mathbf{N}_i$. The right hand side of (14) vanishes for this vector index, and these equations take the form $d_{1E_n-E_1}\langle E_n - E_1, \Lambda\rangle = 0$. This implies (15) for the smallest P. Now let (15) be satisfied for all P preceding some vector $Q \in \mathbf{N}$, and let $\langle Q, \Lambda\rangle \neq 0$. We will show that the right-hand sides of the equations (14) vanish in this case. Since the matrices of the linear parts of systems (8) and (13) possess property b), then $a_{jk} \neq 0$ only if $\lambda_j = \lambda_k$, (that is, $\langle E_j, \Lambda\rangle = \langle E_k, \Lambda\rangle$) but then $\langle Q - E_k + E_j, \Lambda\rangle = \langle Q, \Lambda\rangle \neq 0$ and $d_{jQ-E_k+E_j} = 0$ by our inductive assumption. Therefore the first, third, and fifth summations in the right-hand sides of (14) must vanish. If $P + R = Q$, the product $d_{iP}\tilde{g}_{jR} = 0$, since $\langle P, \Lambda\rangle + \langle R, \Lambda\rangle = \langle Q, \Lambda\rangle \neq 0$; i.e., at least one summation in the last sum is non-zero and, consequently, at least one of the factors in the product $d_{iP}\tilde{g}_{jR}$ is zero. Therefore the second and fourth summations in (14) vanish. It remains to be shown that the coefficient of $y_i Y^Q$ in the series

$$y_i d_i g_i(y_1 d_1, \ldots, y_n d_n) \equiv y_i d_i \sum_S g_{iS} Y^S d_1^{s_1} \ldots d_n^{s_n}$$

$$\equiv y_i \sum_S g_{iS} Y^S d_1^{t_1} \ldots d_n^{t_n} \ , \qquad t_j \geqslant 0 \ , \qquad i = 1, \ldots, n \ .$$

vanishes. From the last expression, it should be obvious that this coefficient is a sum of all terms such as

$$g_{iS} d_{1P_1} \ldots d_{nP_r} \ , \tag{16}$$

for which $S + P_1 + \cdots + P_r = Q$. Since $\langle Q, \Lambda\rangle \neq 0$ and $\langle S, \Lambda\rangle = 0$, then $\langle P_j, \Lambda\rangle \neq 0$ for at least one P_j. In product (16) all the vectors P_j precede Q and, according to our inductive assumption, the corresponding factor in (16) vanishes. This means that the term $\{\psi_i\}_Q$ also vanishes, and we obtain $d_{iQ}\langle Q, \Lambda\rangle = 0$. Hence, $d_{iQ} = 0$ if $\langle Q, \Lambda\rangle \neq 0$. □

Example 1. The following question arises in connection with Theorem 2. Is it possible to simplify further the normal form by means of the transformations of theorem 2? We show here that it is not always possible.

The system

$$\begin{aligned} \dot{x}_i &= x_i \ , \qquad i = 1, \ldots, n-1 \ , \\ \dot{x}_n &= m x_n + \psi(x_1, \ldots, x_{n-1}) \ , \end{aligned} \tag{17}$$

where ψ is a homogeneous polynomial of degree $m > 1$, is a normal form. Here $\Lambda = (1, 1, \ldots, 1, m)$, and the equation

$$\langle Q, \Lambda\rangle \equiv q_1 + \cdots + q_{n-1} + q_n m = 0 \tag{18}$$

has exactly two kinds of solutions $Q \in \mathbf{N}$:

I. $q_n = 0$, $Q = E_i - E_j$, $i, j = 1, \ldots, n-1$;

II. $q_n = -1$, $q_1 + q_2 + \cdots + q_{n-1} = m$, $q_1, \ldots, q_{n-1} \geqslant 0$.

Writing system (17) in form (8):

$$\dot{x}_i = 1x_i \ , \qquad i = 1, \ldots, n-1 \ ,$$

$$\dot{x}_n = x_n \left(m + \sum_{q_1 + \cdots + q_{n-1} = m} g_{n(q_1, \ldots, q_{n-1}, -1)} x_1^{q_1} \ldots x_{n-1}^{q_{n-1}} x_n^{-1} \right) ,$$

we see that (18) is satisfied. According to theorem 2, all transformations which carry system (11) into other normal forms have the form

$$\begin{aligned} x_i &= \sum_{j=1}^{n-1} c_{ij} y_j \ , \qquad i = 1, \ldots, n-1 \ , \\ x_n &= c y_n + \xi(y_1, \ldots, y_{n-1}) \ , \end{aligned} \tag{19}$$

where $c \det(c_{ij}) \neq 0$ and ξ is a homogeneous polynomial of degree m. Under such a transformation system (17) becomes the system

$$\begin{aligned} \dot{y}_i &= y_i \ , \qquad i = 1, \ldots, n-1 \ , \\ \dot{y}_n &= c^{-1}\dot{x}_n - c^{-1}\dot{\xi} = c^{-1} m x_n + c^{-1}\psi - c^{-1} m \xi \\ &= m y_n + c^{-1} m \xi + c^{-1}\psi(x_1, \ldots, x_{n-1}) - c^{-1} m \xi \\ &= m y_n + \tilde{\psi}(y_1, \ldots, y_{n-1}) \ . \end{aligned} \tag{20}$$

We have used here Euler's formula for the homogeneous function ξ. System (20) differs from (17) only in the homogeneous polynomial

$$\tilde{\psi}(y_1, \ldots, y_{n-1}) = c^{-1}\psi(x_1, \ldots, x_{n-1}) \ ,$$

which is obtained from the polynomial ψ by the linear transformation

$$x_i = c^{-1/m} \sum_{j=1}^{n-1} c_{ij} y_j \ , \qquad i = 1, \ldots, n-1 \ .$$

Hence, the problem of determining the complete system of invariants of system (17) with respect to formal transformations reduces to the problem of finding the complete system of invariants of an $(n-1)$-dimensional form of degree m with respect to linear transformations. The same question arises in the reduction to canonical form. Considering how little is known about algebraic invariants in the general case (see Gurevič , 1948), theorem 1 appears to be the best way to obtain general results on the simplification of system (8) with formal transformations. This is why we introduced normal forms, which give neither a full classification of systems nor a complete system of invariants; their normality appears in the normality of the vectors Q and Λ.

By constructing similar examples it is possible to show that the problem of determining the invariants and the classification of formal systems in the class

of formal transformations includes fundamental problems of the theory of algebraic invariants. But in some special cases it is possible to obtain further simplifications of the normal form, and even to find all of its invariants. This will be treated in § 2 of this chapter.

1.5. Distinguished Transformations

As theorem 1 states, the normalizing transformation (7) and the normal form (9) are uniquely determined only after the choice of the resonant coefficients h_{iQ} (for $\langle Q, \Lambda \rangle = 0$) in the normalizing transformation. In particular, all of the resonant coefficients h_{iQ} can be set equal to zero. Such a normalizing transformation (9) is unique; we will call it *distinguished*. It turns out that many properties of the original system are carried over into the normal form if the resonant coefficients are chosen in the proper manner. There will be many examples of this in this chapter. Here, let us show that the "triangularity" of a system can be preserved under a normalizing transformation.

Theorem 3. *Let the series $\varphi_1, \ldots, \varphi_m$ in system* (8) *depend only on $x_1, \ldots, x_m$ ($m \leqslant n$). Then the series $\xi_1, \ldots, \xi_m$ in the distinguished normalizing transformation* (9) *are independent of $y_{m+1}, \ldots, y_n$, and the transformation*

$$x_i = y_i + \xi_i(y_1, \ldots, y_m) , \qquad i = 1, \ldots, m$$

takes the subsystem

$$\dot{x}_i = \lambda_i x_i + \sum_{i>j} a_{ij} x_j + \varphi_i(x_1, \ldots, x_m) , \qquad i = 1, \ldots, m$$

into normal form.

Proof. The proof is a continuation of the proof of theorem 1. The condition of the theorem implies that

$$f_{iQ} = 0 , \qquad \text{if } |q_{m+1}| + \cdots + |q_n| \neq 0 , \qquad i = 1, \ldots, m ;$$

$$h_{iQ} = 0 , \qquad \text{if } \langle Q, \Lambda \rangle = 0 , \qquad i = 1, \ldots, m .$$

We must show that $h_{iP} = g_{iP} = 0$, if $|p_{m+1}| + \cdots + |p_n| \neq 0, i = 1, \ldots, m$. Let this be true for all P which precede the vector Q. We will show this for all Q such that $|q_{m+1}| + \cdots + |q_n| \neq 0$. We will successively show that all summations in the right-hand side of equation (11) vanish for $i \leqslant m$.

1) Since $i \leqslant m$, then the vector $P = Q - E_j + E_i$ has

$$p_k = q_k , \qquad k = m + 1, \ldots, n ,$$

and, by our inductive assumption $h_{iQ-E_j+E_i} = 0$.

2) $P + R = Q$, whence $|p_{m+1}| + \cdots + |p_n| + |r_{m+1}| + \cdots + |r_n| \geqslant |q_{m+1}| + \cdots + |q_n| > 0$; then either $|p_{m+1}| + \cdots + |p_n| \neq 0$ or $|r_{m+1}| + \cdots + |r_n| \neq 0$, or both. In any case, $h_{iP}g_{iR} = 0$ by our inductive assumption.

3) For $j \leqslant m$, just as in 1), $h_{jQ-E_k+E_j} = 0$. For $j > m$, the jth component of the vector $P = Q - E_k + E_j$ is $p_j = q_j + 1$. If $p_j \neq 0$, then $|p_{m+1}| + \cdots + |p_n| \neq 0$, and by our inductive assumption $h_{iQ-E_k+E_j} = 0$. Thus, $(q_j + 1)h_{iQ-E_k+E_j}$ will always vanish for $j = 1, \ldots, n$.

4) See 2). If $p_{m+1} = \cdots = p_n = 0$ then $\sum_{k=1}^{n} p_k g_{kR} = \sum_{k=1}^{m} p_k g_{kR}$, but then $|r_{m+1}| + \cdots + |r_n| \neq 0$ and $g_{1R} = \cdots = g_{mR} = 0$ by our inductive assumption. And if $|p_{m+1}| + \cdots + |p_n| \neq 0$, then $h_{iP} = 0$.

5) See 1).

6) We are left with a sum of terms of the form

$$\{(1 + h_i)f_{iS}Y^S(1 + h_1)^{s_1}\ldots(1 + h_n)^{s_n}\}_Q \ . \tag{20'}$$

If $|s_{m+1}| + \cdots + |s_n| \neq 0$, then $f_{iS} = 0$ by the condition of the theorem, and such terms vanish. If $s_{m+1} = \cdots = s_n = 0$, then (20′) is

$$\{(1 + h_i)f_{iS}Y^S(1 + h_1)^{s_1}\ldots(1 + h_m)^{s_m}\}_Q \ ,$$

which is a sum of terms of the form

$$f_{iS}h_{j_1P^{(1)}}\ldots h_{j_kP^{(k)}} \ , \qquad j_1, \ldots, j_k \leqslant m \ ,$$

where $Q = S + P^{(1)} + \cdots + P^{(k)}$. Hence,

$$|q_{m+1}| + \cdots + |q_n| \leqslant \sum_{l=1}^{k} (|p_{m+1}^{(l)}| + \cdots + |p_n^{(l)}|) \ .$$

Therefore $|p_{m+1}^{(l)}| + \cdots + |p_n^{(l)}| \neq 0$ for at least one $P^{(l)}$, and $h_{j_lP^{(l)}}$ vanishes by inductive assumption; that is, the terms (20′) vanish. Consequently, all the terms in $\{(1 + h_i)f_i\}_Q$ are equal to zero. Thus, we have shown that

$$g_{iQ} + h_{iQ}\langle Q, \Lambda\rangle = 0 \ , \qquad i = 1, \ldots, m \ .$$

If $\langle Q, \Lambda\rangle \neq 0$ then $g_{iQ} = 0$ by the definition of the normal form, and we see that $h_{iQ} = 0$. If $\langle Q, \Lambda\rangle = 0$, then $h_{iQ} = 0$ by the condition of the theorem, which implies that $g_{iQ} = 0$. The induction begins with the linear terms, for which the assertion of the theorem is obvious. □

Remark. At the same time, we have proven a slightly broader statement than that of the theorem. Specifically: Let the series $\varphi_1, \ldots, \varphi_m$ in the situation of theorem 1 depend only on $x_1, \ldots, x_m$, and let the resonant part of the normalizing transformation have similar properties (i.e., the resonant coefficients h_{iQ} vanish if $i \leqslant m$ and $q_{m+1} + \cdots + q_n > 0$). Then the series $\xi_1, \ldots, \xi_m$ and $\psi_1, \ldots, \psi_m$ are independent of $y_{m+1}, \ldots, y_n$.

1.6. Small Parameters

Suppose for the system

$$\dot{x}_i = \varphi_i(\underset{\sim}{E}, X) \ , \qquad i = 1, \ldots, n \tag{21}$$

with parameters $\underset{\sim}{E} = (\varepsilon_1, \ldots, \varepsilon_m)$, that the point $\underset{\sim}{E} = 0$, $X = 0$ is a stationary point. Let $\Lambda = (\lambda_1, \ldots, \lambda_n)$ be the vector of eigenvalues of the matrix $\partial\Phi/\partial X$, evaluated at $\underset{\sim}{E} = 0$, $X = 0$, and let the functions φ_i be analytic at this point, i.e., they may be expanded into convergent power series in $\underset{\sim}{E}$ and X.

Theorem 4. *There exists a coordinate change*

$$x_i = \xi_i(\underset{\sim}{E}, Y) \ , \qquad i = 1, \ldots, n \ , \tag{22}$$

invertible at the point $\underset{\sim}{E} = 0$, $Y = 0$, which takes system (21) *into the normal form*

$$\dot{y}_i = \psi_i(\underset{\sim}{E}, Y) = y_i \sum_{\langle Q, \Lambda \rangle = 0} g_{iQ}(\underset{\sim}{E}) Y^Q \ , \qquad i, \ldots, n \ . \tag{23}$$

Here the ξ_i and ψ_i are formal power series in $\underset{\sim}{E}$ and Y, and the g_{iQ} are power series in $\underset{\sim}{E}$.

Proof. Let us consider the small parameters as variables and add to system (21) the system

$$\dot{\varepsilon}_j = 0 \ , \qquad j = 1, \ldots, m \ . \tag{24}$$

In this context, the ε_j can be conveniently treated as variables preceding the variables x_i. For the combined systems (24) and (21), the eigenvalues of the matrix of linear terms evaluated at $\underset{\sim}{E} = 0$, $X = 0$ are $0, \ldots, 0, \lambda_1, \ldots, \lambda_n$. We can explicitly write the linear part of the combined system:

$$\dot{\underset{\sim}{E}} = 0 \ , \qquad \dot{X} = \Pi \underset{\sim}{E} + AX + \cdots \ , \tag{25}$$

where the rectangular $(n \times m)$-matrix Π and the square $(n \times n)$-matrix A both have fixed coefficients. A linear transformation in the X coordinates alone:

$$X = BZ \tag{26}$$

can be used to put the matrix A into Jordan form $J = B^{-1}AB$. We can index the Z variables in such a way that all vanishing eigenvalues $\lambda_1 = \cdots = \lambda_l = 0$ appear first on the main diagonal of J, followed by the non-zero eigenvalues $\lambda_{l+1}, \ldots, \lambda_n$. Then $J = \{J_0, J_1\}$ is a block-diagonal matrix in which all vanishing diagonal elements are found in J_0, and all non-vanishing diagonal elements are in J_1. If we split the vector Z into two vectors, $Z_0 = (z_1, \ldots, z_l)$ and $Z_1 = (z_{l+1}, \ldots, z_n)$, then system (25) takes the form

$$\dot{\underset{\sim}{E}} = 0 \ ,$$

$$\dot{Z}_0 = \Pi_0 \underset{\sim}{E} + J_0 Z_0 + \cdots \ , \tag{27}$$

$$\dot{Z}_1 = \Pi_1 \underset{\sim}{E} + J_1 Z_1 + \cdots = J_1 (J_1^{-1} \Pi_1 \underset{\sim}{E} + Z_1) + \cdots$$

If we let

$$W_0 = Z_0 \ , \qquad W_1 = J_1^{-1} \Pi_1 \underset{\sim}{E} + Z_1 \ . \tag{28}$$

then system (27) takes the form:

$$\dot{\underset{\sim}{E}} = 0 \ ,$$

$$\dot{W}_0 = \Pi_0 \underset{\sim}{E} + J_0 W_0 + \cdots \ ,$$

$$\dot{W}_1 = J_1 W_1 + \cdots$$

Recall that all the eigenvalues of J_0 are equal to zero. The matrix of the linear part of this system has properties a) and b) (of section 1.2), and hence theorem 1 applies to this system in $\underset{\sim}{E}$ and W. By that theorem, there exists an invertible formal change of coordinates

$$\underset{\sim}{E}, W \to \Delta, Y \ , \tag{29}$$

under which system (24), (21) takes on the normal form,

$$\dot{\delta}_j = \delta_j \sum_{\langle Q, \Lambda \rangle = 0} d_{jQ}(\Delta) Y^Q \ , \qquad j = 1, \ldots, m \ ; \tag{30}$$

$$\dot{y}_i = y_i \sum_{\langle Q, \Lambda \rangle = 0} g_{iQ}(\Delta) Y^Q \ , \qquad i = 1, \ldots, n \ . \tag{31}$$

Such a normalizing transformation is not unique. In particular, theorem 3 tells us that there always exists a distinguished normalizing transformation, for which $\delta_j = \varepsilon_j$, $j = 1, \ldots, m$. Under this transformation, system (30) is identical to system (24), and system (31) reduces to system (23). The successive application of transformations (26), (28), and (29) does not change the parameters $\underset{\sim}{E}$ and carries system (21) into the form of (23). □

Example 2. Let the right-hand sides of system (21) be linear in X:

$$\Phi(\underset{\sim}{E}, X) = A(\underset{\sim}{E}) X + \tilde{A}(\underset{\sim}{E}) \ .$$

It is easy to show that the normalizing change of coordinates (22) will likewise be linear in Y:

$$X = B(\underset{\sim}{E}) Y + \tilde{B}(\underset{\sim}{E})$$

as will be the normal form (23):

$$\dot{Y} = C(\underset{\sim}{E}) Y + \tilde{C}(\underset{\sim}{E}) \ ,$$

where $C(\underset{\sim}{E}) = B^{-1}(\underset{\sim}{E})A(\underset{\sim}{E})B(\underset{\sim}{E})$. By theorem 4, the element $c_{ij}(\underset{\sim}{E})$ will be nonzero only if $\lambda_i = \lambda_j$; that is, the matrix $A(\underset{\sim}{E})$ is similar to the block-diagonal matrix $C(\underset{\sim}{E})$ in which every eigenvalue λ_j of the matrix $A(0)$ corresponds to one block. At the same time, further simplifications inside each block can be obtained by similarity transformations (see Arnol'd, 1971a). But these further simplifications are not covered by the theorem on the reduction to normal form. Instead, they correspond to a secondary normalization (see § 2, chapter III).

Theorem 4 was formulated and proved in an article by Bruno [1974b], where one can find further examples of the application of this theorem. In general, if we treat small parameters as local coordinates, then the various specialized perturbation methods are just variations on the reduction of system (21) to normal form (23) (see §§ 3 and 4 of this chapter, as well as Bruno, 1973d, 1979).

1.7. On the Calculation of the Normal Form

Many properties of a given system are more easily investigated with the help of its normal form. For such purposes, it is usually sufficient to establish general properties of the normal form and to calculate a few of its lower-order terms. To perform these calculations, it is first necessary to make a linear coordinate transformation under which the matrix of the linear part is put into triangular form with the property b) (for example, into Jordan form; see Gantmacher, 1959). Then we must perform a non-linear change of coordinates in accordance with theorem 1. The calculations of the coefficients of the normal form can be carried out in the same order as in the proof of theorem 1.

Generally speaking, these calculations are quite cumbersome. The coefficients g_{iQ} and h_{iQ} are polynomials of the coefficients f_{jP} and δ_P, where

$$\delta_P = \begin{cases} 0 \ , & \text{if } \langle P, \Lambda \rangle \neq 0 \ , \\ 1 \ , & \text{if } \langle P, \Lambda \rangle = 0 \ . \end{cases}$$

It would be convenient to write these polynomials out explicitly for small values of n and $\|Q\|$. Starzhinsky [1972, 1973, 1974, 1977] did so in part; however, his notation was rather inconvenient, for, it does not facilitate calculations, but complicates them and sometimes is the cause of errors. In the same work he solved the question of stability for certain mechanical problems with the help of the normal forms. But if system (1) has already been normalized in its lowest-order terms, (i.e. all the lower-order terms in the right-hand sides are resonant), then the calculation of further terms of the normal form can be simplified.

Theorem 5. *For system* (8), *let there exist some vector K, such that for all terms $f_{iQ}X^Q$ in* (8), *including the linear terms, we have*:

1) $\langle Q, K \rangle \geqslant 0$,

2) $f_{iQ} = 0$, *if* $\langle Q, K \rangle < s$ *and* $\langle Q, \Lambda \rangle \neq 0$—*that is, system* (8) *has been normalized up to terms of degree s (with respect to vector K).*

Then in the normal form (9), *we will have*

$$g_{iQ} = f_{iQ} \ , \qquad \text{if } \langle Q, K\rangle < 2s \ . \tag{32}$$

Proof. The proof proceeds exactly as did the proof of theorem 1. We choose as our normalizing transformation the distinguished transformation for which $h_{iQ} = 0$ when $\langle Q, \Lambda\rangle = 0$. By induction on the ordering of the indices Q, we will show that

$$h_{iP} = 0 \text{ and } g_{iP} = f_{iP} \ , \qquad \text{if } \langle P, K\rangle < s \ . \tag{33}$$

Let this be true for all vector indices P which precede a vector Q: $\langle Q, K\rangle < s$. In equation (11), the right-hand side is a sum of terms of the form

$$g_{iR}h_{iP} \text{ and } f_{iS}h_{j_1P_1}\dots h_{j_kP_k} \ , \tag{34}$$

where all the vectors P, R, S, P_1, ..., P_k precede Q. In agreement with our condition, $\langle S, K\rangle \geqslant 0$ and, by our inductive assumption, $\langle R, K\rangle \geqslant 0$. Since

$$\langle Q, K\rangle = \langle R, K\rangle + \langle P, K\rangle = \langle S, K\rangle + \langle P_1, K\rangle + \dots + \langle P_k, K\rangle < s \ ,$$

then $\langle P, K\rangle < s$ and $\langle P_l, K\rangle < s$ for at least one l in $l = 1, \dots, k$. By the inductive assumption, the corresponding coefficient h_{iP} or $h_{j_lP_l}$ must vanish. Consequently,

$$g_{iQ} + \langle Q, \Lambda\rangle h_{iQ} = f_{iQ} \ .$$

If $\langle Q, \Lambda\rangle \neq 0$, then $f_{iQ} = 0$ by the conditions of the theorem, and $h_{iQ} = 0$. If $\langle Q, \Lambda\rangle = 0$, then $g_{iQ} = f_{iQ}$. The induction begins with the linear terms, for which the assertion is trivally satisfied. Thus, property (33) is proved.

We now use induction with respect to the order of the vector indices to prove property (32). Let this property be satisfied for all those vector indices which precede the vector Q: $\langle Q, K\rangle < 2s$, $\langle Q, \Lambda\rangle = 0$. We consider the terms (34) from the right-hand side of equation (11). Since $\langle R, \Lambda\rangle = 0$, then $\langle P, \Lambda\rangle = 0$. But then $h_{iP} = 0$, since we use the distinguished normalizing transformation. We have shown for these terms (34) that

$$\langle S, K\rangle \geqslant 0 \ , \qquad \langle P_l, K\rangle \geqslant s \ ,$$

but

$$2s > \langle Q, K\rangle = \langle S, K\rangle + \langle P_1, K\rangle + \dots + \langle P_k, K\rangle \geqslant sk \ ;$$

consequently, $k = 1$, and we are left with terms of the form $f_{iS}h_{j_1P_1}$ and f_{iQ}, where $S + P_1 = Q$. By property (33), $\langle P_1, K\rangle \geqslant s$, so that $\langle S, K\rangle = \langle Q, K\rangle - \langle P_1, K\rangle < 2s - s = s$. By the condition of the theorem, f_{iS} is then non-zero only if $\langle S, \Lambda\rangle = 0$. But then $\langle P_1, \Lambda\rangle = 0$, and $h_{j_1P_1} = 0$ since the transformation is distinguished. We thus obtain equation (32). The induction begins with the linear terms, for which the assertion of the theorem is trivially true, and proceeds from there. □

Remark. In the situation of theorem 5, if the matrix A is diagonal and the f_i contain no terms of degree less than s, then

$$g_{iQ} = \sum_{P+R=Q} \frac{1}{\langle P, \Lambda\rangle} f_{iR}\langle R, F_P\rangle + f_{iQ} \ , \tag{35}$$

where $s \leqslant \langle P, K\rangle$, $\langle R, K\rangle < 2s \leqslant \langle Q, K\rangle < 3s$; $\langle Q, \Lambda\rangle = 0$, $\langle P, \Lambda\rangle \neq 0$, and $\langle R, \Lambda\rangle \neq 0$.

Proof. It is evident from equation (11) that

$$g_{iQ} = \{\varphi_i\}_Q \ ,$$

where $\{\varphi_i\}_Q$ is the coefficient of $y_i Y^Q$ in the series $\varphi_i(Y + \Xi(Y))$. Let us calculate the contribution to this coefficient from the term $x_i f_{iR} X^R$. We have

$$\begin{aligned} x_i f_{iR} X^R &= y_i Y^R f_{iR}(1 + h_i)(1 + h_1)^{r_1} \ldots (1 + h_n)^{r_n} \\ &= y_i Y^R f_{iR}(1 + h_i + \langle R, H\rangle + \cdots) \ , \end{aligned}$$

where the dots indicate terms of degree greater than $2s$, since by (33) the expansions of h_j begin with terms of degree s. Since $\langle R, K\rangle > s$, we have the following expression for terms of degree less than $3s$:

$$\{\varphi_i\}_Q = \sum_{P+R=Q} f_{iR}(h_{iP} + \langle R, H_P\rangle) + f_{iQ} \ ,$$

and for terms of degree less than $2s$, we have $\{\varphi_i\}_P = f_{iP}$. Thus, given the above bounds on the vector exponents, we can obtain from formula (11) the equation

$$g_{iQ} = \sum_{P+R=Q} f_{iR}(h_{iP} + \langle R, H_P\rangle) + f_{iQ} \ , \qquad h_{iP}\langle P, \Lambda\rangle = f_{iP} \ .$$

Since $P + R = Q$ and $\langle Q, \Lambda\rangle = 0$, then $\langle P, \Lambda\rangle = -\langle R, \Lambda\rangle$. Hence

$$f_{iR} h_{iP} = f_{iP} f_{iR} \langle P, \Lambda\rangle^{-1} = -f_{iP} h_{iR} \ ,$$

from which (35) follows immediately. □

Example 3. If $K = (1, \ldots, 1)$, then $\langle Q, K\rangle = \|Q\|$, which is just the usual degree of the term $f_Q X^Q$. Theorem 5 asserts that in a system normalized to degree s, the calculation of the coefficients of the normal form up to degree $2s$ reduces to isolating the resonant terms (see lemma 3§4 in Bruno, 1971, 1972a). If the matrix A is diagonal, then $s = 1$, and theorem 5 and the remark following tell us that $g_{iQ} = f_{iQ}$ for $\|Q\| = 1$, $\langle Q, \Lambda\rangle = 0$; moreover, formula (32) is true for $\|Q\| = 1$, while formula (35) applies for $\|Q\| = 2$ and $\|P\| = \|R\| = 1$. In order to employ this formula, we need (for each Q: $Q \in \mathbf{N}$, $\|Q\| = 2$, $\langle Q, \Lambda\rangle = 0$) to find all possible choices of P and R such that $P + R = Q$, P, $R \in \mathbf{N}$, and $\|P\| = \|R\| = 1$. In particular, for oscillating systems with neutral linear part, this formula helps us

calculate the cubic terms of the normal form, which determine the stability and other asymptotic properties of solutions of a given system.

For example, let $n = 2$ and $\lambda_1 = -\lambda_2$, the only vector $Q \in \mathbf{N}$ with $\|Q\| = 2$ and $\langle Q, \Lambda \rangle = 0$ is $Q = (1, 1)$. Possible values of $R, P \in \mathbf{N}$ are

R	$(2, -1)$	$(1,0)$	$(0,1)$	$(-1,2)$
P	$(-1,2)$	$(0,1)$	$(1,0)$	$(2,-1)$

Formula (35) takes the form

$$g_{i(1,1)} = -\frac{2}{3\lambda_1} f_{i(2,-1)} f_{1(-1,2)} - \frac{1}{\lambda_1} f_{i(1,0)} f_{1(0,1)} + \frac{1}{\lambda_1} f_{i(0,1)} f_{2(1,0)}$$
$$+ \frac{2}{3\lambda_1} f_{i(-1,2)} f_{2(2,-1)} + f_{i(1,1)} \,.$$

Hence,

$$g_{1(1,1)} = f_{1(1,1)} + \frac{2}{3\lambda_1} f_{1(-1,2)} f_{2(2,-1)} - \frac{1}{\lambda_1} f_{1(1,0)} f_{1(0,1)} + \frac{1}{\lambda_1} f_{1(0,1)} f_{2(1,0)} \,.$$

If λ_1 is pure imaginary, then $\operatorname{Re} g_{1(1,1)}$ is the Lyapunov number; the stationary point is stable for $\operatorname{Re} g_{1(1,1)} < 0$ and unstable for $\operatorname{Re} g_{1(1,1)} > 0$ (compare with § 1 of chapter II). In agreement with section 1.9 of chapter II, for a given real system in complex notation, the reality condition $f_{1(p,q)} = \bar{f}_{2(q,p)}$ is satisfied. Therefore,

$$\operatorname{Re} g_{1(1,1)} = \operatorname{Re} f_{1(1,1)} - \operatorname{Re}\{1/\lambda_1 f_{1(1,0)} f_{1(0,1)}\}$$
$$= \operatorname{Re} f_{1(1,1)} + \operatorname{Im}\{f_{1(1,0)} f_{1(0,1)}\}/\operatorname{Im}\lambda_1 \,.$$

Let $\mathbf{D}$ be some set of vectors in n-dimensional real space $\mathbf{R}_1^n$. Let $\mathbf{D}_+$ denote the set consisting of finite sums of vectors of the set $\mathbf{D}$. If $\mathbf{C}$ is another set of points, then we let $\mathbf{C} + \mathbf{D}$ denote the set of sums of a vector from $\mathbf{C}$ and a vector from $\mathbf{D}$.

We rewrite system (8) (including the linear terms) in the form

$$\dot{x}_i = x_i f_i = x_i \sum f_{iQ} X^Q \,, \qquad i = 1, \ldots, n \,;$$

we let $F_Q = (f_{1Q}, \ldots, f_{nQ})$ and $F = \sum F_Q X^Q$. Let us denote by $\mathbf{D}(F)$ the *support* of F—that is, the set of points Q for which $F_Q \neq 0$.

Theorem 5′. *In the situation of theorem* 1 *with a distinguished normalizing transformation* (9), *the coefficients h_{iQ} and g_{iQ} depend on a coefficient f_{jP} only when $Q \in P + \mathbf{D}_+(F)$.*

The proof is not given here; it is similar to the proof of lemma 2 in § 7 of Bruno [1971, 1972a]. It is not difficult to derive theorem 5 as a consequence of theorem 5′. In theorem 5′, the normalizing transformation need not be distin-

guished. Let $H_\Lambda = (h_{1\Lambda}, \ldots, h_{n\Lambda})$ be the resonant parts of the series h_i. Then the coefficients h_{iQ} and g_{iQ} depend on the coefficient f_{jP} only when

$$Q \in P + \mathbf{D}_+(F) + \mathbf{D}_+(H_\Lambda) .$$

1.8. Invariance

In the rest of this part of the chapter, we present the results found in Bruno [1975b]. It will be convenient to slightly alter our notation. We will write a given system in the form

$$\dot{X} = AX + \Theta(X) . \tag{36}$$

Let the matrix J be the Jordan normal form of A; then $J = B^{-1}AB$. If we apply the linear coordinate change

$$X = BZ \tag{37}$$

system (36) becomes

$$\dot{Z} = JZ + \Phi(Z) , \tag{38}$$

where the Taylor series of the function $\Phi(Z)$ has no constant or linear terms. Now we consider a formal, non-linear change of coordinates of the form

$$Z = W + \Xi(W) , \tag{39}$$

where $\Xi(W) = (\xi_1, \ldots, \xi_n)$ is a power series in W with neither constant nor linear terms. Let transformation (39) carry system (38) into the formal system

$$\dot{W} = JW + \Psi(W) , \tag{40}$$

where $\Psi = (\psi_1, \ldots, \psi_n)$ is a power series with no constant or linear terms. We denote by $\Lambda = (\lambda_1, \ldots, \lambda_n)$ the vector of the diagonal elements of J. By the Λ-*resonant part* of the series Ξ, we mean the series $\Xi_\Lambda = (\xi_{1\Lambda}, \ldots, \xi_{n\Lambda})$:

$$\xi_{j\Lambda} = w_j \sum h_{iQ} W^Q ,$$

where the summation is taken over all Q for which

$$\langle Q, \Lambda \rangle \equiv q_1\lambda_1 + \cdots + q_n\lambda_n = 0 . \tag{41}$$

System (40) will be a normal form if the series Ψ coincides with its own Λ-resonant part Ψ_Λ. According to theorem 1, every system (38) can be taken into normal form (40) by means of a formal change of coordinates (39). The coefficients of the series Ξ_Λ may be chosen arbitrarily, but these determine the full series Ψ and Ξ uniquely. Note that the series Φ, Ξ, and Ψ are related by the partial differential equations

$$\Psi(W) + \frac{\partial \Xi}{\partial W}(JW + \Psi(W)) = J\Xi(W) + \Phi(W + \Xi(W)) , \tag{42}$$

and theorem 1 becomes a statement about the properties of formal solutions Ξ and Ψ of this system of equations.

Let a system similar to (38)

$$\dot{\tilde{Z}} = \tilde{J}\tilde{Z} + \tilde{\Phi}(\tilde{Z}) \tag{43}$$

be transformed by a coordinate change

$$\tilde{Z} = \tilde{W} + \tilde{\Xi}(\tilde{W}) \tag{44}$$

into the system

$$\dot{\tilde{W}} = \tilde{J}\tilde{W} + \tilde{\Psi}(\tilde{W}) \ . \tag{45}$$

Let $\tilde{\Lambda}$ be the vector of the diagonal elements of the Jordan matrix $\tilde{J}$. If $\tilde{\Lambda} \neq \Lambda$, then the Λ-resonant part Ξ_Λ of some series Ξ and its $\tilde{\Lambda}$-resonant part $\Xi_{\tilde{\Lambda}}$ may contain terms with different exponents. Let us denote by $\bar{\Xi}(W)$ the series with coefficients that are the complex conjugates of the coefficients of the series $\Xi(W)$.

Theorem 6. *Let system* (38) *be reduced to normal form* (40) *by transformation* (39). *In the following three cases we can obtain for system* (43) *the normalizing transformation* (44) *and the normal form* (45) *according to the following formulae*:

1) *If* $\tilde{J} = \bar{J}$, *and* $\tilde{\Phi}(Z) = \bar{\Phi}(Z)$, *then* $\tilde{\Xi}(W) = \bar{\Xi}(W)$, $\tilde{\Psi}(W) = \bar{\Psi}(W)$, *and* $\tilde{\Xi}_{\tilde{\Lambda}}(W) = \bar{\Xi}_\Lambda(W)$;

2) *If* $\tilde{J} = \delta J$, $\tilde{\Phi}(Z) = \delta\Phi(Z)$ *for some number* $\delta \neq 0$ *then* $\tilde{\Xi}(W) = \Xi(W)$, $\tilde{\Psi}(W) = \delta\Psi(W)$, *and* $\tilde{\Xi}_{\tilde{\Lambda}}(W) = \Xi_\Lambda(W)$;

3) *If, for some non-singular matrix* K, $\tilde{J} = K^{-1}JK$ *and* $\tilde{\Phi}(Z) = K^{-1}\Phi(KZ)$, *then* $\tilde{\Xi}(W) = K^{-1}\Xi(KW)$, $\tilde{\Psi}(W) = K^{-1}\Psi(KW)$, *and* $\tilde{\Xi}_{\tilde{\Lambda}}(W) = K^{-1}\Xi_\Lambda(KW)$.

Proof. The proof in each case follows this scheme. First, we prove that transformation (44) takes system (43) into system (45); that is, we show that the system of formal equations

$$\tilde{\Psi}(\tilde{W}) + \frac{\partial\tilde{\Xi}}{\partial\tilde{W}}(\tilde{J}\tilde{W} + \tilde{\Psi}(\tilde{W})) = \tilde{J}\tilde{\Xi}(\tilde{W}) + \tilde{\Phi}(\tilde{W} + \tilde{\Xi}(\tilde{W})) \tag{46}$$

is satisfied. Then we show that $\tilde{\Psi}_{\tilde{\Lambda}}(\tilde{W}) = \tilde{\Psi}(\tilde{W})$ (i.e., system (45) is a normal form) and that the series $\tilde{\Xi}_{\tilde{\Lambda}}(W)$ is related to $\Xi_\Lambda(W)$ in the manner suggested by the theorem. We proceed with the separate cases:

1) If we replace all the coefficients in system (42) with their complex conjugates, then the equality still holds. Consequently, system (46) is satisfied for $\tilde{\Xi}(\tilde{W}) = \bar{\Xi}(\tilde{W})$ and $\tilde{\Psi}(\tilde{W}) = \bar{\Psi}(\tilde{W})$. In this case, $\tilde{\Lambda} = \bar{\Lambda}$ and the set of real solutions Q of equation (41) is the same as for the equation

$$\langle Q, \tilde{\Lambda}\rangle = 0 \ . \tag{47}$$

Consequently, for every vector power series $\Xi(W)$, the resonant parts Ξ_Λ and $\Xi_{\tilde{\Lambda}}$ are identical. Hence the series $\tilde{\Psi}$ consists solely of resonant terms and $\tilde{\Xi}_{\tilde{\Lambda}} = \bar{\Xi}_\Lambda$.

2) Multiplying the equations (42) by δ, we find that the series $\tilde{\Xi}(\tilde{W}) = \Xi(\tilde{W})$ and $\tilde{\Psi}(\tilde{W}) = \delta\Psi(\tilde{W})$ are solutions of system (46). Since $\tilde{\Lambda} = \delta\Lambda$, the sets of real solutions of equations (41) and (47) are identical. Consequently, for every vector power series $\Xi(W)$ we have $\Xi_\Lambda = \Xi_{\tilde{\Lambda}}$. Therefore the series $\delta\Psi$ consists solely of resonant terms, and $\tilde{\Xi}_{\tilde{\Lambda}} = \Xi_\Lambda$.

3) If we make the coordinate change $W = K\tilde{W}$ in system (42) and then multiply the system on the left by the matrix K^{-1}, the equality is preserved. System (46) therefore has solutions $\tilde{\Xi}(\tilde{W}) = K^{-1}\Xi(K\tilde{W})$, $\tilde{\Psi}(\tilde{W}) = K^{-1}\Psi(K\tilde{W})$. Let $\Xi(W)$ be some vector power series and let $\tilde{\Xi}(\tilde{W}) = K^{-1}\Xi(K\tilde{W})$. We will now show that

$$\tilde{\Xi}_{\tilde{\Lambda}}(\tilde{W}) = K^{-1}\Xi_\Lambda(K\tilde{W}) \ . \tag{48}$$

Let L and $\tilde{L}$ be the diagonal matrices formed from the diagonal elements of the Jordan matrices J and $\tilde{J}$, respectively; that is, $L = \operatorname{diag}\Lambda$ and $\tilde{L} = \operatorname{diag}\tilde{\Lambda}$. By the conditions of the theorem, $JK = K\tilde{J}$ $(K = (k_{ij}))$. According to Gantmacher [1959, ch. VIII, § 1, thm. 1], $k_{ij} = 0$ if $\lambda_i \neq \tilde{\lambda}_j$. Hence,

$$LK = K\tilde{L} \ . \tag{49}$$

Let $\underset{\sim}{Z}(W)$ and $\underset{\sim}{H}(W)$ be vector fields. Their commutator (or Poisson bracket) is the vector field

$$\Theta(W) = [\underset{\sim}{Z}, \underset{\sim}{H}] = \sum_{j=1}^{n} \left(\frac{\partial \underset{\sim}{Z}}{\partial w_j}\eta_j - \frac{\partial \underset{\sim}{H}}{\partial w_j}\zeta_j \right) .$$

The commutator is linearly dependent on each of the fields $\underset{\sim}{Z}$ and $\underset{\sim}{H}$, and is invariant under a change of coordinates. For example, if $W = K\tilde{W}$, then

$$K^{-1}\Theta(K\tilde{W}) = [K^{-1}\underset{\sim}{Z}(K\tilde{W}), K^{-1}\underset{\sim}{H}(K\tilde{W})] \ , \tag{50}$$

where the commutator is calculated in the $\tilde{W}$ coordinates. Note that for $\zeta_i = w_i f_i W^Q$, $i = 1, \ldots, n$, where $(f_1, \ldots, f_n)$ is a fixed vector, we have $[\underset{\sim}{Z}, LW] = \underset{\sim}{Z}\langle Q, \Lambda\rangle$. Therefore the resonant part $\underset{\sim}{Z}_\Lambda$ of a series $\underset{\sim}{Z}$ is distinguished by the property $[\underset{\sim}{Z}_\Lambda, LW] = 0$. By (49) and (50)

$$[\underset{\sim}{Z}, LW] = [K^{-1}\underset{\sim}{Z}(K\tilde{W}), K^{-1}LK\tilde{W}] = [K^{-1}\underset{\sim}{Z}(K\tilde{W}), L\tilde{W}] \ .$$

That is, the $\tilde{\Lambda}$-resonant terms of the vector series $K^{-1}\underset{\sim}{Z}(K\tilde{W})$ correspond to the Λ-resonant terms of the series $\underset{\sim}{Z}$, but non-$\tilde{\Lambda}$-resonant terms of the series $K^{-1}\underset{\sim}{Z}(K\tilde{W})$ correspond to non-Λ-resonant terms of the series $\underset{\sim}{Z}$, and conversely. Consequently, the resonant part $\tilde{\Xi}_{\tilde{\Lambda}}$ corresponds to the resonant part Ξ_Λ, and statement (48) is satisfied.

The following equations are implied by (48),

$$\tilde{\Psi}_{\tilde{\Lambda}}(\tilde{W}) = K^{-1}\Psi_\Lambda(K\tilde{W}) = K^{-1}\Psi(K\tilde{W}) = \tilde{\Psi}(\tilde{W}) \ ,$$

$$\tilde{\Xi}_{\tilde{\Lambda}}(\tilde{W}) = K^{-1}\Xi_\Lambda(K\tilde{W}) \ ,$$

that is, system (45) is a normal form, and the resonant part of the normalizing transformation (39) determines the resonant part of the normalizing transformation (44). The theorem is proved. □

1.9. Real Systems

Suppose that system (36) is real; that is,

$$\bar{A} = A \ , \qquad \bar{\Theta}(X) = \Theta(X) \ . \tag{51}$$

Then for the Jordan form J we can choose a block-diagonal matrix

$$J = \{J_1, J_2, J_3\} \tag{52}$$

where $J_1 = \bar{J}_2$ is a Jordan matrix of order l and $J_3 = \bar{J}_3$ is a Jordan matrix of order m. Clearly, $2l + m = n$. The eigenvalues $\lambda_1, \ldots, \lambda_n$ form the diagonal elements of J, with $\lambda_i = \bar{\lambda}_{l+i}$ $(i = 1, \ldots, l)$ as the complex eigenvalues and $\lambda_{2l+1}, \ldots, \lambda_n$ as the real eigenvalues. The sub-diagonal elements are either zeros or some real number σ, which can be chosen arbitrarily. We denote the $k \times k$ identity matrix by $E^{(k)}$.

Lemma 1. *If the matrix A is real, then there exists a transformation* (37) *to system* (38) *with matrix* (52) *such that*

$$\bar{B} = BI \ , \tag{53}$$

where I is the block matrix

$$I = \begin{pmatrix} 0 & E^{(l)} & 0 \\ E^{(l)} & 0 & 0 \\ 0 & 0 & E^{(m)} \end{pmatrix} . \tag{54}$$

The proof follows immediately from theorem 28 in § 34 of Pontryagin's text [1961], according to which there exists a basis, in which the matrix A has a Jordan form, which differs from the complex conjugate basis only by a permutation. The matrix I gives this permutation for the Jordan form (52) (see also Hartman, 1964, Ch. 4, § 9). □

Note that

$$I^2 = E^{(n)} \ , \qquad I^{-1} = I \ , \qquad \bar{I} = I \ , \qquad \bar{J} = IJI \ . \tag{55}$$

We now introduce the matrix

$$C = \begin{pmatrix} E^{(l)} & iE^{(l)} & 0 \\ E^{(l)} & -iE^{(l)} & 0 \\ 0 & 0 & E^{(m)} \end{pmatrix} . \tag{56}$$

for which by (54)

$$\bar{C} = IC \ . \tag{57}$$

Therefore, and by virtue of (53) and (55), the matrix $D = BC$ must be real, since $\bar{D} = \overline{BC} = BIIC = BC = D$. Consequently, the transformation $X = DY$ is real. Here, the real coordinates Y are related to the complex coordinates Z by the *standard transformation*

$$Z = CY \ . \tag{58}$$

This means that if the X coordinates are real, then so are the Y coordinates, while the Z coordinates satisfy *the reality relation* $\bar{Z} = IZ$ (see Hartman, 1964, Ch. 6, § 9). Note that in system (38), $\Phi(Z) = B^{-1}\Theta(BZ)$. Therefore, and according to (51) and (53) we must have $I^{-1}\bar{\Phi}(IZ) = I^{-1}B^{-1}\bar{\Theta}(BIZ) = B^{-1}\Theta(BZ) = \Phi(Z)$, i.e.,

$$I^{-1}\bar{\Phi}(IZ) = \Phi(Z) \ . \tag{59}$$

Now let system (38) be transformed into system (40) by the transformation (39). We perform the standard transformation $W = CV$ with matrix (56). We will show that under the condition

$$I^{-1}\bar{\Xi}(IW) = \Xi(W) \tag{60}$$

the transformation from Y to V is real. In fact, this transformation is $Y = V + C^{-1}\Xi(CV)$. By (57), (55), and (60),

$$\overline{C^{-1}\Xi(CV)} = \bar{C}^{-1}\bar{\Xi}(\bar{C}V) = C^{-1}I\bar{\Xi}(ICV) = C^{-1}\Xi(CV) \ .$$

Theorem 7. *Let system* (38) *be transformed into the normal form* (40) *by the coordinate change* (39), *and let property* (59) *be satisfied. Then property* (60) *will also be satisfied, if the resonant part* $\Xi_A(W)$ *of the series* $\Xi(W)$ *is such that*

$$I^{-1}\bar{\Xi}_A(IW) = \Xi_A(W) \ . \tag{61}$$

In particular, the property (60) *is satisfied for* $\Xi_A = 0$.

Proof. The proof consists of the repeated use of theorem 6. Since $I^{-1}JI = \bar{J}$ is a Jordan matrix, then the third statement of theorem 6 with $K = I$ applies to system (38). Applying as well the first statement of that theorem and the property $\bar{I} = I$, we find that system (43), where $\tilde{J} = I^{-1}\bar{J}I$ and $\tilde{\Phi}(Z) = I^{-1}\bar{\Phi}(IZ)$, is transformed into the normal form (45) by transformation (44), where $\tilde{\Xi}(W) = I^{-1}\bar{\Xi}(IW)$. In addition, $\tilde{\Xi}_A(W) = I^{-1}\bar{\Xi}_A(IW)$ and $\tilde{\Psi}(W) = I^{-1}\bar{\Psi}(IW)$. But by assumption, properties (59) and (61) are satisfied. Hence $\tilde{J} = J$, $\tilde{\Phi} = \Phi$, and $\tilde{\Xi}_A = \Xi_A$; that is, systems (38) and (43) coincide, as do the resonant parts of transformations (39) and (44). By theorem 1, such normalizing transformations coincide, and

$$\Psi(W) = I^{-1}\bar{\Psi}(IW) \ , \tag{62}$$

that is, (60) is satisfied. The theorem is proved. □

This proof is a generalization of a proof given by Siegel [1956, § 14] for a more specific class of systems.

Example 4. Let $n = 2$. If the eigenvalues λ_1 and λ_2 are real, then $I = E^{(2)}$, and system (38) and its normal form (40) are both real systems. If the eigenvalues are pure imaginary, then $\lambda_1 = \bar{\lambda}_2 = -\lambda_2$, and $I = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. The reality condition (59) for system (38) is $\varphi_1(z_1, z_2) = \bar{\varphi}_2(z_2, z_1)$. By theorem 7, the normal form (40) can be chosen such that an analogous relationship (62) for Ψ is satisfied.

1.10. Linear Automorphisms

Suppose that, under the linear transformation

$$X = M\tilde{X} \ , \qquad t = \delta\tilde{t} \ , \tag{63}$$

where M is a non-singular matrix and the number δ is non-zero, system (36) is transformed into

$$d\tilde{X}/d\tilde{t} = \tilde{A}\tilde{X} + \tilde{\Theta}(\tilde{X}) \ . \tag{64}$$

If, under the change of variables, $\tilde{X} \to X, \tilde{t} \to t$ given by (63), system (64) becomes system (36) (that is $\tilde{A} = A$ and $\tilde{\Theta}(X) = \Theta(X)$), then transformation (63) is called a *linear automorphism* of system (36). If system (36) has an automorphism (63), then system (38), obtained from (36) with transformation (37), has an automorphism

$$Z = N\tilde{Z} \ , \qquad t = \delta\tilde{t} \ , \tag{65}$$

where $N = B^{-1}MB$; that is,

$$\delta N^{-1}JN = J \ , \tag{66}$$

$$\delta N^{-1}\Phi(NZ) = \Phi(Z) \ . \tag{67}$$

Theorem 8. *Let the coordinate change* (39) *transform system* (38) *into the normal form* (40), *and let system* (38) *have an automorphism* (65). *Then the normal form* (40) *has the automorphism* $W = N\tilde{W}, t = \delta\tilde{t}$, *if, in the normalizing transformation* (39),

$$N^{-1}\Xi_A(NW) = \Xi_A(W) \ . \tag{68}$$

In particular, this is true for $\Xi_A = 0$.

Proof. We apply the second assertion of theorem 6 to system (38); to the resulting system, we apply the third assertion of that theorem with $K = N$ (applicable as a consequence of (66)). We thus find that the system

$$\dot{Z} = \delta N^{-1}JNZ + \delta N^{-1}\Phi(NZ) \tag{69}$$

under the transformation

$$Z = W + N^{-1}\Xi(NW) \tag{70}$$

becomes the normal form

$$\dot{W} = \delta N^{-1} J N W + \delta N^{-1} \Psi(NW) \ . \tag{71}$$

Furthermore,

$$\{N^{-1} \Xi(NW)\}_\Lambda = N^{-1} \Xi_\Lambda(NW) \ . \tag{72}$$

By assumptions (66) and (67), the original systems (69) and (38) coincide; by assumption (68), and by virtue of (72), the resonant parts of the normalizing transformations are the same. By theorem 1, the normalizing transformations (70) and (39) also coincide, as do the normal forms (71) and (40). The theorem is proved. □

Example 5. Let system (38) be invariant under the exchange of $-z_n$ for z_n; that is, the series φ_n contains only odd powers of z_n, while the remaining φ_i contain only even powers of z_n. Then the normal form (40) can be chosen such that it is invariant with respect to the exchange of w_n and $-w_n$.

§ 2. The Integration and Classification of Normal Forms

2.1. Power Transformations

If all the right-hand sides in the normal form are written as $y_i g_i$:

$$\dot{y}_i = y_i g_i \equiv y_i \sum g_{iQ} Y^Q \ , \tag{1}$$

then $G_0 = \Lambda$ and the coefficient g_{iQ} differs from zero only when $\langle Q, \Lambda \rangle = 0$. This follows, for non-linear terms, from the definition of the normal form. We will show here that it holds for linear terms as well. In fact, the term $a_{ij} y_j$ corresponds, in the notation of (1), to the term $y_i g_{iE_j - E_i} y_j y_i^{-1}$. For this term,

$$Q = E_j - E_i \ , \qquad \langle Q, \Lambda \rangle = \lambda_j - \lambda_i \ ;$$

but by property b) of section 1.2, $\lambda_j - \lambda_i = 0$ if $a_{ij} \neq 0$.

Notation (1) and the definition of the normal form allow a simple geometrical interpretation.

We place every non-zero coefficient g_{iQ} in correspondence with a point Q in the integer lattice of n-dimensional real-affine space $\mathbf{R}^n$, where $Q = (q_1, \ldots, q_n)$. We will denote the set of all such points by $\mathbf{D}(g_1, \ldots, g_n)$ or simply $\mathbf{D}(G)$. The normal form differs from an arbitrary system (1) by the fact that its set $\mathbf{D}(G)$ lies entirely in the linear subspace which is orthogonal to the vectors $\operatorname{Re} \Lambda$ and $\operatorname{Im} \Lambda$. This allows us, with the help of a *power transformation*

$$u_i = y_1^{\alpha_{i1}} \ldots y_n^{\alpha_{in}} \ , \qquad i = 1, \ldots, n \tag{2}$$

to reduce the order of the system, which is a normal form, sometimes even to integrate it. Transformations of systems of differential equations by changes of coordinates (2) were treated in detail in § 2 of an earlier article by the author [Bruno, 1965].

Lemma 1. *Let* $\mathbf{D}(G)$ *be a set of points* $Q \in \mathbf{R}^n$ *such that in system* (1). $G_Q = (g_{1Q}, \ldots, g_{nQ}) \neq 0$, *and let transformation* (2), *where the* α_{ij} *are real and, (if* $\alpha = (\alpha_{ij})$) $\det \alpha \neq 0$, *transform system* (1) *into the system*

$$\dot{u}_i = u_i g_i'(U) \equiv u_i \sum_{Q'} g'_{iQ'} U^{Q'} , \qquad i = 1, \ldots, n . \tag{3}$$

Then

$$\mathbf{D}(G') = \alpha^{*-1}\mathbf{D}(G) . \tag{4}$$

Proof. Let us write

$$\ln Y = (\ln y_1, \ldots, \ln y_n) , \qquad \ln U = (\ln u_1, \ldots, \ln u_n) .$$

In vector notation, system (1) takes the form

$$(\ln^{\bullet} Y) = \sum_{Q \in \mathbf{D}(G)} G_Q \exp\langle Q, \ln Y\rangle ,$$

while system (3) is

$$(\ln^{\bullet} U) = \sum_{Q' \in \mathbf{D}(G')} G'_Q \exp\langle Q', \ln U\rangle ,$$

and transformation (2) becomes

$$\ln U = \alpha \ln Y .$$

Under this transformation, the term Y^Q becomes

$$Y^Q \equiv \exp\langle Q, \ln Y\rangle = \exp\langle Q, \alpha^{-1} \ln U\rangle$$
$$= \exp\langle \alpha^{-1*} Q, \ln U\rangle \equiv U^{\alpha^{*-1}Q} ,$$

and

$$(\ln^{\bullet} U) = \alpha(\ln^{\bullet} Y) = \sum_{Q \in \mathbf{D}(G)} \alpha G_Q Y^Q = \sum_{Q \in \mathbf{D}(G)} \alpha G_Q U^{\alpha^{*-1}Q} .$$

Consequently,

$$\sum_{Q \in \mathbf{D}(G)} \alpha G_Q \exp\langle \alpha^{-1*} Q, \ln U\rangle = \sum_{Q' \in \mathbf{D}(G')} G'_{Q'} \exp\langle Q', \ln U\rangle .$$

Therefore,

$$Q' = \alpha^{*-1} Q , \qquad G'_{Q'} = \alpha G_Q . \tag{4'}$$

Thus, the set $\mathbf{D}(G')$ of points Q' such that $G'_{Q'} \neq 0$ can be found from $\mathbf{D}(G)$ by linear transformation (4). This proves the lemma. □

Theorem 1. *Let system* (1) *be a normal form, and let δ be the number of linearly independent points $Q \in \mathbf{N}$ which satisfy the equation $\langle Q, \Lambda\rangle = 0$. There exists a power transformation* (2) (α_{ij} *integers*, $\det \alpha = \pm 1$), *under which the normal form* (1) *becomes the system*

$$\dot{u}_i = u_i g_i'(u_1, \ldots, u_\delta) , \qquad i = 1, \ldots, n . \tag{5}$$

The first δ equations of this system define a system of order δ, while the remaining equations reduce to quadratures.

Proof. The proof relies on lemma 1 and on certain properties of free Abelian groups with a finite number of generators (see Kurosh, 1953). Let $\mathbf{K}$ be a linear subspace in $\mathbf{R}^n$ which spans the set of solutions $Q \in \mathbf{N}$ of the equation $\langle Q, \Lambda \rangle = 0$. Clearly, $\mathbf{K}$ is of dimension δ. The vectors of the integral lattice in $\mathbf{R}^n$ form, under addition, the free Abelian group $\mathbf{Z}^n$ with n generators. $\mathbf{M} = \mathbf{Z}^n \cap \mathbf{K}$ is a subgroup of $\mathbf{Z}^n$ of rank δ which is, clearly, a free Abelian group together with the factor-group $\mathbf{Z}^n/\mathbf{M}$ of rank $n - \delta$. There exists an isomorphism of the group $\mathbf{Z}^n$ and the direct sum $\mathbf{M} \oplus \mathbf{Z}^n/\mathbf{M}$ which is the identity mapping on $\mathbf{M}$. Let the vectors R_1, $\ldots$, R_δ be generators of $\mathbf{M}$ and let $R_{\delta+1}, \ldots, R_n$ be generators of $\mathbf{Z}^n/\mathbf{M}$. Taken together, they give a full system of generators of $\mathbf{M} \oplus \mathbf{Z}^n/\mathbf{M}$. Their images under the isomorphism described above, $R_1, \ldots, R_\delta, S_{\delta+1}, \ldots, S_n$, give a complete system of generators of $\mathbf{Z}^n$; that is, the matrix α^*, in which the ith column is $R_i (i \leqslant \delta)$ or $S_i (i \geqslant \delta)$, is unimodular, and $\alpha^{*-1} R_i = E_i$. At the same time, the vectors R_i define a linear basis of $\mathbf{K}$. The transformation $Q' = \alpha^{*-1} Q$ carries the subspace $\mathbf{K}$ into the coordinate subspace spanned by $E_1, \ldots, E_\delta$; that is, $q'_{\delta+1} = \cdots = q'_n = 0$.

By lemma 1, systems (1) and (5) and transformation (2) are related by formula (4). But $\mathbf{D}(G) \subset \mathbf{K}$, so that $\mathbf{D}(G') \subset \alpha^{*-1}\mathbf{K}$; that is $g'_1, \ldots, g'_n$ are independent of $u_{\delta+1}, \ldots, u_n$. The proof is complete. □

For another proof of theorem 1 which gives a way to calculate the matrix α see Bruno [1965].

2.2. Classification

Until now, we have been indifferent to the ordering of the Jordan blocks in the matrix $(\partial \varphi_i / \partial x_j)_0$ or the order of the eigenvalues λ_i in the vector Λ. It turns out that the normal form sometimes has nearly triangular form under the right ordering of the Jordan blocks or, equivalently, the correct indexing of variables.

In what follows, we will distinguish two cases for the eigenvalues $\lambda_1, \ldots, \lambda_n$. Let these numbers be considered as points in the complex plane.

Case 1: There exists a line, passing through the origin in the complex plane such that none of the points λ_i lie on one side of the line, while exactly l (counting multiplicity) of the points λ_i lie on the line. The variables are indexed so that these l values are the first l components of Λ.

Case 2: Points λ_i lie on both sides of *any* line through the origin.

Let $\mathfrak{N}$ be the convex hull of the points $\lambda_1, \ldots, \lambda_n$; $\mathfrak{N}$ may be a point, a line segment, or a polygon. Clearly, in case 2 $\mathfrak{N}$ is a polygon which encloses the origin.

We identify four subcases of case 1:

a) The origin lies outside of $\mathfrak{N}$; then there exists a line $\mathfrak{M}$ through the origin such that all of the points λ_i lie to one side of the line $\mathfrak{M}$, and no points λ_i lie on $\mathfrak{M}(l = 0)$;

b) the origin is a vertex of the polygon $\mathfrak{N}$ or an endpoint of the line segment $\mathfrak{N}$. Then $\lambda_1 = \cdots = \lambda_l = 0$ $(0 < l < n)$, the remaining λ_i are non-zero, and there exists a line $\mathfrak{M}$ through the origin such that $\lambda_{l+1}, \ldots, \lambda_n$ lie on one side of $\mathfrak{M}$;

c) $\mathfrak{N}$ is a polygon, and the origin lies on one of its edges but not at a vertex; let $\lambda_1, \dots, \lambda_l$ be all the points λ_i that lie along this edge $(1 < l < n)$; then the line $\mathfrak{M}$, along this edge, is such that $\lambda_1, \dots, \lambda_l$ lie on $\mathfrak{M}$, while $\lambda_{l+1}, \dots, \lambda_n$ lie on one side of $\mathfrak{M}$;

d) $\mathfrak{N}$ is a line segment which contains the origin in its interior; then the line $\mathfrak{M}$ which lies along this line segment is such that all of the λ_i lie on $\mathfrak{M}$ (i.e., $l = n$).

In the following discussion, we will assume that in case 1 we will always choose the line $\mathfrak{M}$ described above, that l is the number of points λ_i on $\mathfrak{M}$, and that the variables Y have been so indexed that

$$0 = \mu_1 = \cdots = \mu_l < \mu_{l+1} \leqslant \cdots \leqslant \mu_n \ , \tag{6}$$

where μ_i is the distance of the point λ_i from the line $\mathfrak{M}$. Condition (6) determines the relative placement in the matrix J of Jordan blocks with different μ_i, leaving arbitrary the relative placement of those blocks with equal μ_i. We note that the line $\mathfrak{M}$ is uniquely defined in subcases 1c) and 1d). In subcases 1a) and 1b), we can choose for $\mathfrak{M}$ any line which lies inside of some pair of vertical angles with vertex at the origin. In particular, we can choose $\mathfrak{M}$ such that $\mu_i = \mu_j$ only if $\lambda_i = \lambda_j$.

Let us denote by τ a complex number of unit modulus such that the two-vector $(\operatorname{Re}\tau, \operatorname{Im}\tau)$ is orthogonal to $\mathfrak{M}$ and τ lies on the same side of $\mathfrak{M}$ as $\mathfrak{N}$. We let

$$\mu_j = \operatorname{Re}(\bar{\tau}\lambda_j) \ , \qquad \nu_j = \operatorname{Im}(\bar{\tau}\lambda_j) \ , \qquad j = 1, \dots, n$$

and

$$\underset{\sim}{M} = (\mu_1, \dots, \mu_n) \ , \qquad \underset{\sim}{N} = (\nu_1, \dots, \nu_n) \ .$$

Then

$$\bar{\tau}\Lambda = \underset{\sim}{M} + i\underset{\sim}{N} \quad (\text{here } i^2 = -1) \ .$$

It is easy to see that μ_j is, as before, the distance of the point λ_j from the line $\mathfrak{M}$. It is convenient to imagine that $\mathfrak{M}$ is the imaginary axis and that $\mathfrak{N}$ lies to its right. Then $\tau = 1$, $\underset{\sim}{M} = \operatorname{Re}\Lambda$, $\underset{\sim}{N} = \operatorname{Im}\Lambda$. We note that the equation

$$\langle Q, \Lambda \rangle = 0 \tag{7}$$

is, for real Q, equivalent to the system of two equations

$$\langle Q, \underset{\sim}{M} \rangle = 0 \ , \qquad \langle Q, \underset{\sim}{N} \rangle = 0 \ . \tag{8}$$

Let $V = (v_1, \dots, v_n)$ be an n-vector; let us write $V' = (v_1, \dots, v_l)$ and $V'' = (v_{l+1}, \dots, v_n)$. By $V \geqslant 0$, we mean $v_1 \geqslant 0, \dots, v_n \geqslant 0$; we will use this notation widely. For example, condition (6) implies that $\underset{\sim}{M}' = 0$, $\underset{\sim}{M}'' > 0$.

Lemma 2. *If Λ belongs to case* 1, *then equation* (8) *has only those solutions $Q \in \mathbf{N}_i$ for which $Q'' = 0$ if $i \leqslant l$; if $i > l$, then either*

$$Q'' = E_j - E_i \ , \qquad \text{where } \mu_j = \mu_i \ , \tag{9}$$

or

$$Q'' = \sum_{j=l+1}^{m} q_j E_j - E_i \ , \qquad \text{where } q_j \geqslant 0 \ , \qquad m < i \ , \qquad \mu_m < \mu_i \ , \tag{10}$$

Proof. In consequence of (6), equation (8) is equivalent to the equation

$$\langle Q'', \underset{\sim}{M}'' \rangle = 0 \ . \tag{11}$$

Suppose $Q \in \mathbf{N}_i$, $i \leqslant l$; then $Q'' \geqslant 0$ and $\langle Q, \underset{\sim}{M} \rangle > 0$ if $Q'' \neq 0$. Consequently, for such solutions of equation (8), $Q'' = 0$.

Now suppose that $Q \in \mathbf{N}_i$, $i > l$; then equation (11) has, by virtue of (6), solutions of only two types:

I. $\|Q''\| = 0$, $Q'' = E_j - E_i$, if $\mu_j = \mu_i$.

II. $\|Q''\| > 0$, $q_i = -1$, $q_{i+1} = \cdots = q_n = 0$.

These are solutions (9) and (10). □

Corollary. *If Λ belongs to case* 1, *then equation* (7) *has only the kinds of solutions described in Lemma* 2.

Theorem 2. *If Λ belongs to case* 1, *then the normal form has the form* (12):

$$\dot{y}_i = \psi_i \ , \qquad i = 1, \ldots, l \ , \tag{12$'$}$$

$$\dot{y}_i = \sum_{j=l+1}^{n} b_{ij} y_j + \sum b_{iq_{l+1}\ldots q_{i-1}} y_{l+1}^{q_{l+1}} \ldots y_{i-1}^{q_{i-1}} \ , \quad i = l+1, \ldots, n \ . \tag{12$''$}$$

Here ψ_i, b_{ij}, and $b_{iq_{l+1}\ldots q_{i-1}}$ are power series in $y_1, \ldots, y_l$. The first sum in (12″) *is taken over those $j > l$ for which equation* (9) *is satisfied; the second sum is taken over all integers $q_{l+1}, \ldots, q_{i-1}$ for which* (10) *is satisfied.*

Proof. The proof follows immediately from lemma 2. In the normal form (1), $g_{iQ} \neq 0$ only if $Q \in \mathbf{N}_i$ and $\langle Q, \Lambda \rangle = 0$. If $i \leqslant l$, then, by lemma 2, $q_{l+1} = \cdots = q_n = 0$; that is, $y_{l+1}, \ldots, y_n$ do not appear in ψ_i in non-zero powers. Consequently, ψ_i is independent of $y_{l+1}, \ldots, y_n$.

If $i > l$, then Q'' has the form of either (9) or (10). The corresponding terms are exactly those that appear in (12″). □

Remark. If Λ belongs to case 1, then subsystem (12′) of the normal form (12) is itself a normal form of order l. Analogously to theorem 2, it can be deduced from lemma 2 and theorem 2 of § 1 that, in case 1, a transformation from normal form (12) to another normal form takes the form (13):

$$y_i = \eta_i(z_1, \ldots, z_i) \ , \qquad i = 1, \ldots, l \ , \tag{13$'$}$$

$$y_i = \sum_{j=l+1}^{n} c_{ij}(Z') z_j + \eta_i(z_1, \ldots, z_{i-1}) \ , \qquad i = l+1, \ldots, n \ . \tag{13$''$}$$

Moreover, (13′) takes the l-th order normal form (12′) into another normal form of order l, while the η_i in (13″) contain no terms of less than second degree in Z''.

We will show here what theorem 2 means in the subcases of case 1.

1a) Here $l = 0$. Hence, there is no subsystem (12′) in (12), and the series b_{ij} and $b_{iq_{l+1}\dots q_{i-1}}$ are constants. Moreover, $b_{ij} = 0$ for $j > i$, since we have chosen the Jordan matrix of the linear part of (12) to be lower triangular as in theorem 1 of § 1 of this chapter. Thus, the normal form is triangular:

$$\dot{y}_i = \lambda_i y_1 + \sum b_{iq_1\dots q_{i-1}} y_1^{q_1} \dots y_{i-1}^{q_{i-1}}\,, \qquad i = 1, \dots, n\,, \tag{14}$$

where the sum is taken over all integers $q_1, \dots, q_{i-1} \geqslant 0$ for which

$$\lambda_i = q_1\lambda_1 + \dots + q_{i-1}\lambda_{i-1}\,, \qquad i = 1, \dots, n\,. \tag{15}$$

Normal form (14) was found by Dulac [1912]. Equations (15) have only a finite number of solutions for integral, non-negative $q_1, \dots, q_{i-1}$. Hence, the right-hand sides in (14) are polynomials. For the same reason, transformations from one normal form to another are also polynomial transformations and hence they are biregular. The equations of system (14) can, clearly, be successively solved by quadratures.

1b) Here $\lambda_1 = \dots = \lambda_l = 0$, so the first sum in (12″) is taken over those j for which $\lambda_i = \lambda_j$, while the second is taken over integral, non-negative $q_{l+1}, \dots, q_{i-1}$ such that $q_{l+1} + \dots + q_{i-1} > 1$ and $\lambda_i = q_{l+1}\lambda_{l+1} + \dots + q_{i-1}\lambda_{i-1}$.

The integration of normal form (12) reduces in this case to two steps: first, solution of the lth order system and second, the successive integration of systems of linear equations with variable coefficients. If the $\lambda_{l+1}, \dots, \lambda_n$ are all different, then the second step is just the process of successive quadratures of case 1a).

1c) Here, subsystem (12′) is itself a normal form of l-th order for which theorem 1 is applicable. Therefore, integration of system (12) reduces, in this subcase, to the solution of a nonlinear system of order less than l, to an integration up to the order l, and to the successive solution of systems of linear differential equations with variable coefficients. If the distances of the points $\lambda_{l+1}, \dots, \lambda_n$ from the line $\mathfrak{M}$ are all different, then (8) has no solutions $Q'' \neq 0$, and it is not necessary to solve systems of linear differential equations, but only to integrate linear equations successively.

1d) In this case, theorem 2 yields no simplification at all.

Suppose that $\mathbf{L}$ is the linear subspace in $\mathbf{R}_1^n$ defined by the real solutions of equation (7), and that $\mathbf{0} = \{Q \geqslant 0\}$ is the non-negative orthant (quadrant, octant, etc.). The above classification of Λ can be geometrically interpreted (in $\mathbf{R}_1^n$) in the following way:

1a) $\mathbf{L} \cap \mathbf{0} = 0$. That is, if $Q \geqslant 0$ and $Q \neq 0$, then $\langle Q, \Lambda \rangle \neq 0$. In fact, for such vectors Q, the complex number $\langle Q, \Lambda \rangle / \|Q\|$ lies either inside or on the boundary of the convex hull $\mathfrak{N}$ of points λ_i, and hence must be non-zero.

1b) $\mathbf{L} \cap \mathbf{0}$ is an l-dimensional face of the cone $\mathbf{0}$, defined by the equation

$$Q'' = 0\,. \tag{16}$$

1c) $\mathbf{L} \cap \mathbf{0}$ is an $(l-1)$-dimensional subset of an l-dimensional face of $\mathbf{0}$ defined by equation (16).

1d) Here $\mathbf{L}$ has dimension $n-1$ and intersects $\mathbf{0}$ at interior points. If 1a), 1b), and 1c) are considered as degenerate cases, then this is the regular case, since the intersection does not degenerate to part of the boundary of $\mathbf{0}$. Generally speaking, here it is impossible to expect any simplification of the normal form other than the reduction of the order of the normal form by at least one (theorem 1).

2. This case differs from 1d) only in that the dimension of $\mathbf{L}$ is $n-2$.

For more detailed properties of the generalized triangularity of normal forms, see section III, § 2 in Bruno [1971, 1972a].

2.3. One-dimensional Normal Forms

This section follows from the author's article [1973c] which discusses other related questions as well. Let us consider the formal system

$$\begin{aligned} \dot{u}_1 &= u_1 g_1(u_1) , \qquad g_1 = \sum_{k=m}^{\infty} g_{1k} u_1^k , \quad g_{1m} \neq 0 , \quad m > 0 , \\ \dot{u}_2 &= u_2 g_2(u_1) , \qquad g_2 = \sum_{k=0}^{\infty} g_{2k} u_1^k . \end{aligned} \tag{17}$$

If $g_{20} \neq 0$, then this system is a normal form, since $\lambda_1 = 0$, $\lambda_2 = g_{20}$ and the vector exponents have the form $Q = (k, 0)$. According to theorem 2 of § 1, transformations which take normal form (17) into other normal forms have the form

$$\begin{aligned} u_1 &= v_1 d_1(v_1) \equiv v_1 \sum_{k=0}^{\infty} d_{1k} v_1^k , \qquad d_{10} \neq 0 , \\ u_2 &= v_2 d_2(v_1) \equiv v_2 \sum_{k=0}^{\infty} d_{2k} v_1^k , \qquad d_{20} \neq 0 . \end{aligned} \tag{18}$$

Theorem 3. *There exists a formal transformation* (18) *with* $d_{10} = d_{20} = 1$, *which takes system* (17) *into the form*

$$\begin{aligned} \dot{v}_1 &= v_1(\gamma_m v_1^m + \gamma_{2m} v_1^{2m}) , \\ \dot{v}_2 &= v_2 \beta(v_1) \equiv v_2 \sum_{k=0}^{m} \beta_k v_1^k , \end{aligned} \tag{19}$$

where all coefficients are uniquely determined by system (17).

Proof. We begin by deriving the first equation. Let transformation (18.1) take equation (17.1) into the form

$$\dot{v}_1 = v_1 \gamma(v_1) , \qquad \gamma = \sum_{k=m}^{\infty} \gamma_k v^k .$$

Then the series g_1, d_1, and γ are related by the equation

$$\left(d_1 + v_1 \frac{dd_1}{dv_1}\right) v_1 \gamma = v_1 d_1 g_1(v_1 d_1) \equiv g_{1m} v_1^{m+1} d_1^{m+1} + \cdots , \tag{20}$$

and from (20) we obtain an equation for the coefficients of v_1^{m+1+k}:

$$(1 + k) d_{1k} \gamma_m + \gamma_{k+m} = g_{1m}(m + 1) d_{1k} + c_{m+1+k} , \tag{21}$$

where c_{m+1+k} is a polynomial in d_{1j} and γ_{m+j} with $j < k$. For $k = 0$, since $d_{10} = 1$, formula (20) gives $\gamma_m = g_{1m}$; equation (21) takes the form

$$g_{1m}(k - m) d_{1m} + \gamma_{k+m} = c_{m+1+k} \qquad (k > 0) .$$

This equation will be satisfied, if, successively, for $k = 1, 2, \ldots$, we take

$$\gamma_{k+m} = 0 , \qquad d_{1k} = c_{m+1+k} g_{1m}^{-1} (k - m)^{-1} \text{ for } k \neq m ,$$

$$\gamma_{2m} = c_{2m+1} , \qquad d_{1m} = \text{an arbitrary number} .$$

Thus, equation (17.1) takes the form of (19.1) under transformation (18.1); then equation (17.2) takes the form

$$\dot{u}_2 = u_2 \tilde{\beta}(v_1) , \qquad \tilde{\beta} = \sum_{k=0}^{\infty} \tilde{\beta}_k v_1^k . \tag{22}$$

The coefficients $\gamma_m, \gamma_{2m}, \tilde{\beta}_0, \ldots, \tilde{\beta}_m$ do not depend upon the number d_{1m}, i.e., they are uniquely defined. It is easy to verify that the coordinate change

$$u_2 = v_2 \exp \int \left(\sum_{k=m+1}^{\infty} \tilde{\beta}_k v_1^k \right) v_1^{-1} (\gamma_m v_1^m + \gamma_{2m} v_1^{2m})^{-1} \, dv_1 \tag{23}$$

has the form of (18.2) and takes equation (22) into form (19.2), where $\beta_k = \tilde{\beta}_k$. This proves the theorem. □

If we make a time transformation as described by Bruno [1971, 1972a, in example (3c) of the introduction], system (17) with $g_{20} \neq 0$ becomes the system

$$dv_1/dt_1 = v_1^{m+1} , \qquad dv_2/dt_1 = v_2(\beta_0 + \beta_m v_1^m)$$

under a formal change of variables.

It follows from theorem 3 and the results of Horn [1899, 1900, 1913] (see also Bruno, 1971, 1972a, example (3c) in the introduction; Martinet et Ramis, 1982) that the analytic system

$$\dot{x}_1 = x_1 g_1(x_1) , \qquad \dot{x}_2 = x_2 g_2(x_1) + a(x_1) , \qquad g_{20} \neq 0 \tag{24}$$

can, under an analytic change of variables $x_1 = y_1 d_1(y_1)$, $x_2 = y_2 d_2(y_1) + c(y_1)$,

take the form

$$\dot{y}_1 = y_1(\gamma_m y_1^m + \gamma_{2m} y_1^{2m}) \ ,$$

$$\dot{y}_2 = y_2 \sum_{k=0}^{m} \beta_k y_1^k + \sum_{k=1}^{m} b_k y_1^k \ .$$

That is, system (24) has, in the class of analytic transformations, m new invariants, the b_k, in addition to the $m + 3$ formal invariants of γ and β. The number of coefficients appearing in system (19) is $m + 3$; one of these can be set equal to (unity) by a linear transformation $y_1 = \tilde{c}_1 z_1$. Consequently, normal form (17) has $m + 2$ continuous invariants with respect to formal coordinate changes.

Note that the dimension of system (17) is $\delta = 1$. Now consider arbitrary normal forms (1) with $\delta = 1$. That is, all solutions $Q \in \mathbf{N}$ of the equation $\langle Q, \Lambda \rangle = 0$ are proportional to one vector R. We will distinguish two cases:

a) One of the coordinates of the vector R is equal to -1. We can assume that $r_n = -1$. The normal form is

$$\dot{y}_i = \lambda_i y_i \ , \qquad i = 1, \dots, n-1 \ ,$$

$$\dot{y}_n = \lambda_n y_n + a y_1^{r_1} \dots y_{n-1}^{r_{n-1}} \ .$$

The value of the coefficient a is affected only by linear transformations $y_i = c_i \tilde{y}_i$ such that $\tilde{a} = C^R a$. Therefore, in the class of complex transformations a is either 0 or 1; in the class of real transformations the sign of a can be preserved as well (if $a \neq 0$).

b) All the coordinates of the vector R are non-negative ($r_i \geqslant 0$). The examination of this case is the basic aim of this section. We will assume that the integral vector R is the smallest possible, i.e., that the greatest common divisor of its components r_i is 1. Then the normal form (1) is

$$\dot{y}_i = y_i \sum_{k=0}^{\infty} g_{ik} Y^{kR} \ , \qquad i = 1, \dots, n \ . \tag{25}$$

Let us write $G_k = (g_{1k}, \dots, g_{nk})$; clearly $G_0 = \Lambda = (\lambda_1, \dots, \lambda_n)$, so that $\langle R, G_0 \rangle = 0$. Consider first the case $\langle R, G \rangle \not\equiv 0$; since $\langle R, G \rangle \equiv \Sigma \langle R, G_k \rangle Y^{kR}$, then for some natural number m

$$\langle R, G_j \rangle = 0 \text{ for } j < m \ , \qquad \langle R, G_m \rangle \neq 0 \ . \tag{26}$$

Note that, in agreement with theorem 2 of § 1, the transformation taking normal form (25) into another normal form has the form

$$y_i = z_i \sum_{k=0}^{\infty} d_{ik} Z^{kR} \ , \qquad i = 1, \dots, n \ . \tag{27}$$

Theorem 4. *If property* (26) *is satisfied by the formal system* (25), *then there exists a formal, invertible transformation* (27) *with* $d_{i0} = 1$ *which takes system* (25)

into the form

$$\dot{z}_i = z_i\left(\sum_{k=0}^{m} \tilde{g}_{ik} Z^{kR} + ar_i Z^{2mR}\right), \qquad i = 1, \ldots, n, \tag{28}$$

where all coefficients are uniquely determined by system (25).

Proof. Let $S_1 = R$, and let $S_i = (s_{i1}, \ldots, s_{in})$, $i = 2, \ldots, n$, be integral vectors such that the determinant $\det(s_{ij}) = 1$. For the proof of their existence see section 2.1 of this chapter. We make the power transformation

$$u_i = y_1^{s_{i1}} \ldots y_n^{s_{in}}, \qquad i = 1, \ldots, n. \tag{29}$$

Then, in accordance with section 2.1, system (25) becomes the system

$$\dot{u}_i = u_i g_i'(u_1), \qquad i = 1, \ldots, n, \tag{30}$$

where $g_i' = \langle S_i, G\rangle$. In particular, by property (26),

$$g_1' = g_{1m}' u_1^m + \cdots, \qquad g_{1m}' \neq 0.$$

Applying theorem 3 to system (30), we find that there exists a formal transformation

$$u_i = v_i d_i(v_1), \qquad d_i(0) = 1, \qquad i = 1, \ldots, n, \tag{31}$$

which takes system (30) into the form:

$$\begin{aligned} \dot{v}_1 &= v_1(\gamma_{1m} v_1^m + \omega_1 v_1^{2m}), \\ \dot{v}_i &= v_i\left(\sum_{k=0}^{m} \gamma_{ik} v_1^k + \omega_i v_i^{2m}\right), \qquad i = 2, \ldots, n, \end{aligned} \tag{32}$$

where the coefficients ω_i $(i > 1)$ can be arbitrarily chosen, but ω_1 and γ_{ij} are uniquely determined. In order to find the ω_i, we must replace $\sum \beta_k v_1^k$ in (23) with $\sum \tilde{\beta}_k v_1^k - \omega_i v_1^{2m}$. We apply

$$v_i = z_1^{s_{i1}} \ldots z_n^{s_{in}}, \qquad i = 1, \ldots, n, \tag{33}$$

so that system (32) becomes

$$\dot{z}_i = z_i\left(\sum_{k=0}^{m} \tilde{g}_{ik} Z^{kR} + \tilde{\omega}_i Z^{2mR}\right), \qquad i = 1, \ldots, n, \tag{34}$$

where $\tilde{G}_k = \alpha\Gamma_k$, $\tilde{\Omega} = \alpha\Omega$, and $\alpha = (\alpha_{ij}) = (s_{ij})^{-1}$.

In order to obtain $\tilde{\Omega} = Ra$, it is sufficient to assume that $\Omega = \alpha^{-1}R\alpha$. Then system (34) will have the form of (28), where $a = \omega_1/(r_1^2 + \cdots + r_n^2)$, since $\alpha^{-1}R = (s_{ij})R = (\langle S_1, R\rangle, \langle S_2, R\rangle, \ldots, \langle S_n, R\rangle)$ and $S_1 = R$. It remains to be shown that the transformation from Y to Z has the required form. In fact, we can find from

(29), (31), and (33) that

$$y_1 = v_1^{\alpha_{i1}} \dots v_n^{\alpha_{in}} d_1^{\alpha_{i1}}(v_1) \dots d_n^{\alpha_{in}}(v_1) = z_i d_1^{\alpha_{i1}}(Z^R) \dots d_n^{\alpha_{in}}(Z^R) \ .$$

Since the d_i are series in non-negative powers of v_1 with constant terms equal to unity, then a product of arbitrary integral powers of the d_i have the same form. That is, the transformation from Y to Z has the required form. □

Remark. Theorems 3 and 4 leave us free to apply a linear transformation $y_i = c_i \tilde{y}_i$, which can reduce by one the number of continuous invariants. For example, we could require in system (28) that $\langle R, \tilde{G}_m \rangle = \pm 1$. Then the invariants of system (28) are the eigenvalues, the coefficients $\tilde{g}_{ik}$ (related by one condition for each i) and the number a. There are thus $n + m(n-1) + 1 = mn - m + n + 1$ invariants. In the real case, the sign of $\langle R, \tilde{G}_m \rangle$ can also be invariant.

We shall now take up the case where $\langle R, G \rangle \equiv 0$ in the normal form (25), i.e., property (26) does not hold and

$$\langle R, G_j \rangle = 0 \ , \qquad j = 0, 1, 2, \dots \ .$$

In this case transformation (29) takes system (25) to system (30) with $g_1'(u_1) \equiv 0$, i.e.,

$$\begin{aligned} \dot{u}_1 &= 0 \ , \\ \dot{u}_i &= u_i g_i'(u_1) \ , \qquad g_i'(0) \neq 0 \ , \qquad i = 2, 3, \dots, n \ . \end{aligned} \tag{34'}$$

The change of variables (31) carries this system to the system,

$$\begin{aligned} \dot{v}_1 &= 0 \ , \\ \dot{v}_i &= v_i g_i'(v_1 d_1(v_1)) \equiv v_i g_i'(u_1) \ , \qquad i = 2, \dots, n \ . \end{aligned} \tag{34''}$$

Thus, only the one series $d_1(v_1)$ of the n series $d_j(v_1)$ of transformation (31) is effective; the others have no influence on the system (34″). But with the help of just one series only one series can be simplified. If $g_2', \dots, g_n'$ are constants (i.e., independent of u_1), then the series $d_1(v_1)$ has no effect on the problem and $g_i'(v_1 d_1(v_1)) \equiv g_i'(v_1) \equiv g_i'(0)$ in (34″). In this case the normal form (25) has the form

$$\dot{y}_i = \lambda_i y_i \ , \qquad i = 1, \dots, n \ ,$$

i.e., it is zero dimensional and unique. If among the series $g_2'(u_1), \dots, g_n'(u_1)$ at least one is non-constant, then there exists a vector T' such that the series

$$\langle T', G'(u_1) \rangle \equiv \tau(u_1) \equiv \sum_{k=0}^{\infty} \tau_k u_1^k$$

differs from a constant. Let τ_l be the first non-vanishing coefficient in the sequence $\tau_1, \tau_2, \dots$. Then $\tau(u_1) = \tau_0 + \tau_l v_1^l$ if we set

$$v_1 = \sqrt[l]{\frac{\tau - \tau_0}{\tau_l}} = u_1 + \frac{\tau_{l+1}}{l\tau_l} u_1^2 + \cdots ,$$

and we obtain $u_1 = v_1 d_1(v_1)$ by inverting this expansion. Then in system (34″) we have that

$$\langle T', G'(v_1 d_1(v_1)) \rangle = \tau_0 + \tau_l v_1^l .$$

If $n = 2$ then it is sufficient to take $T' = (0, 1)$. In this case in system (34″) we have

$$g_2'(v_1 d_1(v_1)) = g_2'(0) + \tau_l v_1^l ,$$

and the normal form (34″) contains the single invariant τ_l which can be normalized by a linear transformation $\tilde{v}_1 = cv_1$. If $n > 2$, then in place of the vector T' we can take the first non-zero vector G_k' in the expansion

$$G'(u_1) - G'(0) = \sum_{k=1}^{\infty} G_k' u_1^k ,$$

constructed for system (34′). Then for the given choice of the series $v_1 d_1(v_1)$ system (34″) is a normal form in the sense of Belitskiĭ [1979b]. However, for $n > 2$ system (34′) has infinitely many formal invariants. By means of the inverse power transformation (33) we move from system (34″) to the normal form

$$\dot{z}_i = z_i \tilde{g}_i , \qquad i = 1, \ldots, n ,$$

which is associated with system (25) by an invertible formal transformation. Thus, everything claimed for systems (34′) and (34″) also holds for this system.

Thus, even in the case of a simple resonance, if $n > 2$ and one of the normal forms (25) of the original system does not satisfy property (26), then the original system may have infinitely many invariants with respect to invertible formal changes of variables.

2.4. The Secondary Normalization

In the last section, we showed that if $\delta = 1$ the "normal" transformations of theorem 2, § 1, essentially simplify the normal form and are of use in finding all of its formal invariants.

Such a situation is encountered in other cases as well. One example is a Hamiltonian system in the case of resonance (see theorem 2, § 1 in Bruno, 1970b). Another is a system in which the Jordan form of the matrix A of the linear part is non-diagonal. Let us discuss the latter in more detail. According to Belitskiĭ [1975a, 1978, 1979a, 1979b], the normal form can be put in the form

$$\dot{Y} = JY + \Psi(Y) , \tag{35}$$

where the formal series Ψ satisfies the equation

$$\frac{\partial \Psi}{\partial Y} J^* Y = J^* \Psi(Y) . \tag{36}$$

(J^* is the transpose matrix of J.) If we introduce the diagonal matrix $L = \{\lambda_1, \ldots, \lambda_n\}$, then the definition of the usual normal form is written (see Sternberg, 1958)

$$\frac{\partial \Psi}{\partial Y} L Y = L \Psi(Y) \ . \tag{37}$$

If equation (36) is satisfied, so is property (37) and, additionally,

$$\frac{\partial \Psi}{\partial Y}(J^* - L) Y = (J^* - L) \Psi(Y) \ .$$

This additional property, although it gives some simplification to the structure of the non-linear terms, nevertheless, yields no further reduction of the order of the system (as did theorem 1, § 1, and theorem 1, § 2).

Example 1 (see Belitskiĭ, 1975b, 1978, 1979b; Bogdanov, 1976; Sadovskiĭ, 1976). Let $n = 2$, and

$$J = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} .$$

Since $\lambda_1 = \lambda_2 = 0$, every two-dimensional formal system (35) is a normal form. But property (36) is satisfied only for systems of the form

$$\begin{aligned} \dot{y}_1 &= \alpha(y_2) + y_1 \beta(y_2) \ , \\ \dot{y}_2 &= y_1 + y_2 \beta(y_2) \ , \end{aligned} \tag{38}$$

where $\beta(0) = \alpha(0) = d\alpha(0)/dy_2 = 0$. But the "dimension" of this system is still two. Series $\alpha(y_2)$ and $\beta(y_2)$ can be uniquely determined under formal transformations with identical linear parts, but they can be changed by a transformation $y_1 = az_1 + \cdots$, $y_2 = bz_1 + az_2 + \cdots$. For generalizations of (38) see Belitskiĭ [1979a, 1979b], Cushman and Sanders [1985], Sadovskiĭ [1982] and Palamodov [1983].

In accordance with theorem 1, under a power transformation the normal form becomes a system which includes a subsystem of order δ with vanishing eigenvalues. As was shown in the preceding section, if $\delta = 1$ it is possible to calculate all of the formal invariants of this system. But when $\delta > 1$ it is, generally speaking, impossible to find all of the formal invariants. For example, when $\delta = 2$ this is the problem of investigating a non-elementary singular point (see § 3 of chapter II). If $\delta > 1$, a secondary normalization does not yield a substantial simplification of the normal form. The local method approach gives a better perspective here. It consists of dividing the neighborhood of a non-elementary singular point (i.e., when $A = 0$) into parts, and finding the "elementary" normal form for each of those parts. In this context, the class of useful transformations is wider than the class of the usual formal transformations in the form of power series (see chapter IV).

§ 3. Analytic Integral Sets

3.1. Statement of the Problem

We will consider the system

$$\dot{x}_i = \varphi_i(X) \ , \qquad \varphi_i(0) = 0 \ , \qquad i = 1, \ldots, n \ , \tag{1}$$

for which the origin $X = (x_1, \ldots, x_n) = 0$ is a stationary point.

Problem 1: find all invariant analytic sets in the neighborhood of the point $X = 0$ which include that point. Here following Bruno [1975c], we shall briefly present, without proofs, some results fundamental to the solution of this problem.

Instead of the problem 1 we shall solve **Problem 2:** find the sets in which system (1) can be normalized by means of an analytic change of coordinates. In particular, when is the transformation

$$x_i = \xi_i(Y) \ , \qquad i = 1, \ldots, n \tag{2}$$

of system (1) to the normal form

$$\dot{y}_i = \psi_i(Y) = y_i \sum_{\langle Q, \Lambda \rangle = 0} g_{iQ} Y^Q \ , \qquad i = 1, \ldots, n \tag{3}$$

analytic?

The answer to the last question was found by Bruno in [Bruno, 1971, 1972a], while problem 2 was answered by Bruno in [Bruno, 1974c]. In sections 3.2–3.7 we will, to simplify our presentation, assume that the eigenvalues of the system, $\lambda_1, \ldots, \lambda_n$, are pure imaginary, i.e.,

$$\operatorname{Re} \Lambda = 0 \ . \tag{4}$$

3.2. Convergence of the Normalizing Transformation

The normalizing transformation is not often analytic. We can therefore ask: for which normal forms (3) does the analyticity of the normalizing transformation (2) follow from the analyticity of the original system (1)? The definitive answer to this question appears in the works of Bruno [1967, 1971, 1972a], and is contained in the following conditions.

Condition A. *In normal form* (3), $\psi_i = \lambda_i y_i a$, $i = 1, \ldots, n$, *where* $a = a(Y)$ *is some power series.*

Condition ω. *Let* $\omega_k = \min |\langle Q, \Lambda \rangle|$ *for* $Q \in \mathbf{N}$, $\|Q\| < 2^k$, *and* $\langle Q, \Lambda \rangle \neq 0$. *Then the series*

$$\sum_{k=1}^{\infty} 2^{-k} \ln \omega_k$$

converges.

Condition ω is a weak arithmetic condition on the eigenvalues $\lambda_1, \ldots, \lambda_n$, which is almost always satisfied. In contrast, condition A places strict limits on the normal form, since it requires that the n series ψ_i be defined in terms of a single series a.

Theorem 1. *If, for system* (1), *the vector* Λ *satisfies condition* ω *and the normal form* (3) *satisfies condition A, then the normalizing transformation* (2) *is analytic.*

Moreover, it is shown in the author's works cited above that condition A cannot be weakened. That is, if the normal form (3) does not satisfy condition A then there exists an analytic system (1) which is transformed into the normal form (3) by a divergent transformation. The normalizing transformation is thus not analytic, and the analytic normalization of (1) is not possible in any neighborhood of the point $X = 0$.

3.3. On Sets

Let the functions $f_1(X), \ldots, f_s(X)$ be analytic at the origin, and let them vanish there. Then the system of equations

$$f_j(X) = 0\,, \qquad j = 1, \ldots, s \tag{5}$$

defines an *analytic set* $\mathcal{M}$ which contains the point $X = 0$. In the ring of convergent power series, there is an ideal $\mathcal{J}$ with basis $f_1, \ldots, f_s$ which corresponds to the set $\mathcal{M}$. If $f_1, \ldots, f_s$ are formal power series, then we will say that the system of equations (5) defines a *formal set* $\mathcal{M}$, to which there is a corresponding ideal $\mathcal{J}$, with basis $f_1, \ldots, f_s$, in the ring of formal power series. The set $\mathcal{M}$ will be analytic if the ideal $\mathcal{J}$ has a basis of convergent power series.

We will call the set $\mathcal{M}$ a (local) *manifold* if the system of equations (5) can be solved with respect to s of the coordinates $x_1, \ldots, x_n$.

Under a formal invertible coordinate change (2), a one-to-one correspondence is established between formal sets. We will therefore consider the image and pre-image as one set in different coordinate systems. In particular, a manifold remains a manifold in any system of coordinates.

The set $\mathcal{M}$ will be *integral* (or *invariant*) for system (1) if

$$\sum_{k=1}^{n} \frac{\partial f_j}{\partial x_k} \varphi_k(X) \in \mathcal{J} , \qquad j = 1, \ldots, s .$$

Problem 1′: Which formal invariant sets of system (1) are analytic?

We ask this because it is relatively easy to calculate the formal invariant sets for the normal form (3). We need only learn how to determine which of those sets are analytic for the original system (1). Ordinarily, this problem has only been posed for manifolds (see Bruno, 1974a, 1971, 1972a, § 10).

Suppose that, for the system

$$\dot{z}_i = \theta_i(Z) , \qquad i = 1, \ldots, n \tag{6}$$

with $\theta_i(0) = 0$, the matrix of the linear part is triangular, and its diagonal is $\Lambda = (\lambda_1, \ldots, \lambda_n)$. Analogously with the notation of (3), let us write

$$\theta_i = z_i \sum h_{iQ} Z^Q , \qquad i = 1, \ldots, n .$$

We isolate the "resonant parts"

$$\theta_{i\Lambda} = z_i \sum h_{iQ} Z^Q \text{ where } \langle Q, \Lambda \rangle = 0 .$$

Let the formal set

$$\mathcal{M} = \{Z: f_1(Z) = \cdots = f_s(Z) = 0\} \tag{7}$$

be integral for system (6). We will say that system (6) is *normalized on the set (7)* if all the differences $\theta_i - \theta_{i\Lambda}$, $i = 1, \ldots, n$, belong to the ideal $\mathcal{J}$ of set (7). For example, the normal form (3) is normalized on any set.

3.4. The Set $\mathcal{A}$

For the normal form (3), let us define the formal set

$$\mathcal{A} = \{Y: \psi_i = \lambda_i y_i a , \qquad i = 1, \ldots, n\} , \tag{8}$$

where a is a free parameter. We can eliminate a from the equations, and obtain a representation of the set $\mathcal{A}$ in the form of (7). In other words, $\mathcal{A}$ is that set for which condition A is satisfied.

Theorem 2. *If all the eigenvalues* λ_i *of system* (1) *are pairwise commensurable, then the set* $\mathcal{A}$ *is analytic, and there exists an invertible, analytic change of coordinates*

$$x_i = \eta_i(Z) , \qquad i = 1, \ldots, n , \tag{9}$$

transforming system (1) *into system* (6), *which is normalized on the set* $\mathcal{A}$.

If condition A is satisfied, then $\mathscr{A}$ is a full neighborhood of the stationary point and, in accordance with theorem 2, system (1) can be normalized throughout the neighborhood; this also follows from theorem 1.

Example 1. Let system (1) have two variables x_1 and x_2 and a small parameter ε (that is, $\dot{\varepsilon} = 0$); also, let $\lambda_1 = -\lambda_2 = i$. Then, in accordance with theorem 4 of § 1, the normal form is

$$\dot{\varepsilon} = 0 \ ; \qquad \dot{y}_j = y_j g_j(\varepsilon, y_1 y_2) \equiv \psi_j \ , \qquad j = 1, 2 \ .$$

Let us write $\rho = y_1 y_2$. The set

$$\mathscr{A} = \{\varepsilon, y_1, y_2 \colon \psi_j = \lambda_j y_j a, \ j = 1, 2\} = \mathscr{A}^1 \cup \mathscr{A}^2 \cup \mathscr{A}^3 \ ,$$

where

$$\mathscr{A}^1 = \{y_2 = 0\} \ , \qquad \mathscr{A}^2 = \{y_1 = 0\} \ ,$$

$$\mathscr{A}^3 = \{\varepsilon, y_1, y_2 \colon g_1 + g_2 = 0\} \ .$$

$\mathscr{A}^1$ and $\mathscr{A}^2$ are the coordinate axes. By theorem 2, the set $\mathscr{A}$ is analytic. If

$$g_j = \lambda_j + c_j \rho + b_j \varepsilon + \cdots \ ,$$

then the component set $\mathscr{A}^3$ is defined by the equation

$$g_1 + g_2 \equiv (c_1 + c_2)\rho + (b_1 + b_2)\varepsilon + \cdots = 0 \ .$$

In the general case, $b_1 + b_2 \neq 0$, and, by the implicit function theorem, this equation has unique solution $\varepsilon = \varepsilon(\rho)$; that is, the component set $\mathscr{A}^3$ is a manifold. For a given real system (1), it includes periodic solutions; the passage of ε through the origin results in a bifurcation of these solutions (compare with Arnol'd, 1972, § 5.6; Pyartli, 1972; Leontovich and Tareyev, 1972; and Tareyev, 1973). This is a Hopf bifurcation.

3.5. The Set $\mathscr{B}$

In theorem 2, the requirement of pairwise commensurable eigenvalues $\lambda_1, \ldots, \lambda_n$ excludes the appearance of small denominators $\langle Q, \Lambda \rangle$. If these denominators can be arbitrarily small, then the set $\mathscr{A}$ is not necessarily analytic (this was shown for manifolds by Bruno, 1974a, 1974b).

Let $L = \{\lambda_1, \ldots, \lambda_n\}$ be a diagonal matrix. We will examine, on the set $\mathscr{A}$, the matrix

$$B = \frac{\partial \Psi}{\partial Y} - La \ ,$$

where a is the parameter which appears in the equations (8) which define the set $\mathscr{A}$. We define the formal set $\mathscr{B}$ as that subset of $\mathscr{A}$ on which the matrix B is

nilpotent. That is,

$$\mathscr{B} = \{Y\colon Y \in \mathscr{A},\, B^n = 0\} \ .$$

Theorem 3. *If the eigenvalues* $\lambda_1, \ldots, \lambda_n$ *of system* (1) *satisfy condition* ω, *then the set* $\mathscr{B}$ *is analytic, and there exists an invertible analytic transformation* (9) *which takes system* (1) *into a system* (6) *which is normalized on the set* $\mathscr{B}$.

3.6. Properties of the Sets $\mathscr{A}$ *and* $\mathscr{B}$

We assume for simplicity that the normal form (3) is analytic. Let us examine the properties of the solutions of this system in the sets $\mathscr{A}$ and $\mathscr{B}$.

1) All solutions in $\mathrm{Re}\,\mathscr{A}$ are conditionally periodic (including periodic and constant solutions). In fact, the value of the parameter a is constant on every solution, and we have

$$y_i = y_i^0 \exp \lambda_i a t \ , \qquad i = 1, \ldots, n \ . \tag{10}$$

2) Let $\mathscr{K}$ be the coordinate subspace

$$y_{n'+1} = \cdots = y_n = 0 \; (n' \leqslant n) \ .$$

Let us denote by $r = r(\mathscr{K})$ an integer such that among $\lambda_1, \ldots, \lambda_{n'}$ there are r numbers which are linearly independent over the rationals, while any $r + 1$ of the $\lambda_1, \ldots, \lambda_{n'}$ are linearly dependent.

Then any solution (10) on the set $\mathrm{Re}(\mathscr{A} \cap \mathscr{K})$, which does not lie in any smaller coordinate subspace is conditionally periodic with r basic frequencies $(0 \leqslant r \leqslant n')$.

3) For a conditionally periodic solution (10) in the set $\mathscr{A}$, the basis of frequencies is entirely determined by the value of the parameter a. But the eigenvalues can be arbitrary.

4) For a conditionally periodic solution (10) in the set $\mathscr{B}$, the value of the parameter a determines the eigenvalues as well as the basis of frequencies. This property distinguishes the set $\mathscr{B}$ from $\mathscr{A}$.

3.7. A Refinement of Theorem 2

Let us write

$$\tilde{\mathscr{A}} = \bigcup_{r(\mathscr{K}) \leqslant 1} (\mathscr{A} \cap \mathscr{K}) \ ,$$

where the union is taken over all coordinate subspaces $\mathscr{K}$ (in Y coordinates) for which $r(\mathscr{K}) \leqslant 1$.

Theorem 4. *The set* $\tilde{\mathscr{A}}$ *is analytic for system* (1), *and system* (1) *can be normalized on that set by an analytic transformation* (9).

This is, clearly, an extension of theorem 2. We note that all formal families of periodic ($r = 1$) and constant ($r = 0$) solutions are found in the set $\mathscr{A}$. By theorem 4, these families are always analytic. The set $\mathscr{A}$ cannot contain formal families of periodic solutions with $r = 0$ only (see Bruno, 1974c). Thus, under condition ω the analytic set $\mathscr{A} \cup \mathscr{B}$ exists for system (1).

Example 2. We consider the Hamiltonian system with m degrees of freedom:

$$\dot{x}_j = \frac{\partial h}{\partial x_{j+m}} , \qquad \dot{x}_{j+m} = -\frac{\partial h}{\partial x_j} , \qquad j = 1, \ldots, m , \tag{10'}$$

where the Hamiltonian h is analytic at the origin, and its expansion begins with quadratic terms. The eigenvalues come in pairs of pure imaginary numbers with opposite sign:

$$\lambda_j = -\lambda_{j+m} = i\alpha_j , \qquad j = 1, \ldots, m .$$

The normal form (3) is likewise a Hamiltonian system

$$\dot{u}_j = \frac{\partial h}{\partial v_j} , \qquad \dot{v}_j = -\frac{\partial h}{\partial u_j} , \qquad j = 1, \ldots, m , \tag{11}$$

where the series h contains only resonant terms. Let $\alpha_1, \ldots, \alpha_m$ be linearly independent over the rational numbers. Then in the normal form (11) we have that

$$h = h(\rho_1, \ldots, \rho_m) , \qquad \text{where } \rho_j = u_j v_j , \qquad j = 1, \ldots, m .$$

We therefore have

$$\mathscr{A} = \left\{U, V: u_j \frac{\partial h}{\partial \rho_j} = \lambda_j u_j a, \ v_j \frac{\partial h}{\partial \rho_j} = \lambda_j v_j a, \ j = 1, \ldots, m\right\} .$$

Let us examine the set $\mathscr{A}$ in the Cartesian coordinates $\underset{\sim}{P} = (\rho_1, \ldots, \rho_m)$. Then every coordinate subspace (in $\underset{\sim}{P}$) contains one component of the set $\mathscr{A}$ which does not lie in a smaller coordinate subspace. Consequently, the set $\mathscr{A}$ consists of $2^m - 1$ such components; of these, for every $d \leqslant m$, there are exactly $\dfrac{m!}{d!(m-d)!}$ components which lie in d-dimensional subspaces of the $\underset{\sim}{P}$ coordinates, for each of which $r = d$. In particular, there is one component,

$$\mathscr{A}^0 = \left\{\underset{\sim}{P}: \frac{\partial h}{\partial \rho_j} = \lambda_j a, \ j = 1, \ldots, m\right\} ,$$

which is situated outside of the coordinate subspaces. On this component, $r = m$, and $B^2 = 0$. In the general case, the matrix B is not nilpotent on coordinate subspaces, so that $\mathscr{B} = \mathscr{A}^0$. Moreover, the m components

$$\mathcal{A}_i = \{\underset{\sim}{P}: \rho_j = 0, j \neq i\} \ , \qquad i = 1, \ldots, m \ ,$$

are the axes in the $\underset{\sim}{P}$ coordinates and have $r = 1$; taken together, they comprise the set $\tilde{\mathcal{A}}$. By theorem 4, system (10′) has m one-parameter analytic families Re $\mathcal{A}_j$ of periodic solutions (these are the families of Lyapunov, 1935a). If the numbers α_i satisfy condition ω, then by theorem 3 the component $\mathcal{A}^0$ will also be an analytic set. If this component has a real part, then it is a one-parameter family of m-dimensional invariant tori with a basis of frequencies $\alpha_1 a, \ldots, \alpha_m a$ and with zero eigenvalues. As $a \to 1$, the tori of this family shrink into the fixed point $X = 0$.

3.8. The Part of the Spectrum not on the Imaginary Axis

Suppose now that the system

$$\dot{x}_i = \varphi_i(X) \ , \qquad i = 1, \ldots, k + l + m \ , \tag{12}$$

analytic in the neighborhood of the fixed point $X = (x_1, \ldots, x_{k+l+m}) = 0$, has l pure imaginary eigenvalues $(\lambda_1, \ldots, \lambda_l)$, k with negative real parts $(\kappa_1, \ldots, \kappa_k)$, and m with positive real parts $(\mu_1, \ldots, \mu_m)$. Then system (12) has three formal integral manifolds: $\mathcal{W}_-$ (corresponding to $\kappa_1, \ldots, \kappa_k, \lambda_1, \ldots, \lambda_l$), $\mathcal{W}_+$ (corresponding to $\lambda_1, \ldots, \lambda_l, \mu_1, \ldots, \mu_m$), and $\mathcal{V} = \mathcal{W}_- \cap \mathcal{W}_+$ (corresponding to $\lambda_1, \ldots, \lambda_l$).

By means of a formal invertible change of variables $X \to Y, U, V$ system (12) is reduced to *the semi-normal form*

$$\begin{aligned} \dot{Y} &= \Psi(Y) + \check{\Psi}(Y, U, V) \ , \\ \dot{U} &= \Theta(Y, U) + \check{\Theta}(Y, U, V) \ , \\ \dot{V} &= \underset{\sim}{Z}(Y, V) + \check{\underset{\sim}{Z}}(Y, U, V) \ , \end{aligned} \tag{13}$$

where $\check{\Psi}(Y, U, 0) = \check{\Psi}(Y, 0, V) = 0$, $\Theta(Y, 0) = \check{\Theta}(Y, 0, V) = \check{\Theta}(Y, U, 0) = 0$, $\underset{\sim}{Z}(Y, 0) = \check{\underset{\sim}{Z}}(Y, U, 0) = \check{\underset{\sim}{Z}}(Y, 0, V) = 0$. That is, the manifolds $\mathcal{W}_-$, $\mathcal{W}_+$, and $\mathcal{V}$ are the coordinate manifolds $\{V = 0\}$, $\{U = 0\}$ and $\{U = 0, V = 0\}$, respectively. In addition, on the manifold $\mathcal{V}$, system (13) induces the system

$$\dot{Y} = \Psi(Y) \ ,$$

which is a normal form, while on the manifold $\mathcal{W}_-$ system (13) becomes

$$\dot{Y} = \Psi(Y) \ , \qquad \dot{U} = \Theta(Y, U) \ ,$$

which is a semi-normal form in the sense of Bruno [1971, 1972a, §9] (as is the system induced on the manifold $\mathcal{W}_+$).

Now, just as we did above, we will define formal sets $\mathcal{A}$, $\mathcal{B}$, and $\tilde{\mathcal{A}}$ on the manifold $\mathcal{V}$. Theorems 2–4 remain applicable to these sets if we substitute semi-normalization instead of normalization. Moreover, for every formal set $\mathcal{M}$

in the manifold $\mathscr{V}$, we define a set $\mathscr{M}_-$ on the manifold $\mathscr{W}_-$ as a set in Y, U coordinates for which $Y \in \mathscr{M}$ and the U coordinates are arbitrary. Similarly, we define a set $\mathscr{M}_+ \subset \mathscr{W}_+$. It turns out that the sets $\mathscr{A}_-$, $\mathscr{A}_+$, $\mathscr{B}_-$, $\mathscr{B}_+$, $\tilde{\mathscr{A}}_-$, $\tilde{\mathscr{A}}_+$ are analytic whenever the corresponding sets $\mathscr{A}$, $\mathscr{B}$, and $\tilde{\mathscr{A}}$ are (see Bruno, 1974c, 1975a). In the sets Re $\mathscr{A}_-$ and Re $\mathscr{A}_+$ lie solutions which asymptotically approach tori in the set Re $\mathscr{A}$ as $t \to +\infty$ and $t \to -\infty$, respectively. By this means we obtain invariant sets which are fibers on "tendrilled tori".

3.9. The Neighborhood of a Torus

Suppose that for a system of ordinary differential equations there is an irreducible invariant j-dimensional torus $\mathscr{T}$, filled with conditionally periodic solutions ($j \geqslant 0$). Let the system be analytic in some neighborhood of this torus.

Problem 1″: In the neighborhood of the torus $\mathscr{T}$, find all invariant analytic sets containing the torus.

We will assume that the system on the torus and the variational system are reducible. Then the torus $\mathscr{T}$ has a frequency basis and a set of eigenvalues κ_i, λ_i, μ_i.

If $j = 0$, then the torus $\mathscr{T}$ is a fixed point; the corresponding problem was discussed in previous sections.

If the given torus $\mathscr{T}$ is not a fixed point ($j > 0$), then it is either a periodic solution ($j = 1$) or an irreducible torus with conditionally periodic solutions ($j > 1$).

In the first case, the variational system is always reducible, and eigenvalues exist. Therefore, normal and semi-normal forms exist, and the constructions of sets $\mathscr{A}$, $\mathscr{B}$, and $\tilde{\mathscr{A}}$ are easily extended. The corresponding extensions of theorems 3 and 4 are also valid (see Bruno, 1974c, 1975a). In particular, every formal family of periodic solutions which adjoins the given periodic solution $\mathscr{T}$ is analytic.

In the second case, the reducibility of the system on the torus $\mathscr{T}$ and the reducibility of the variational system are additional restrictions (on topological obstacles see Bylov, *et al*, 1977). After the frequency basis and eigenvalues are found, the construction of the semi-normal form and the selection of the sets $\mathscr{A}$ and $\mathscr{B}$ can proceed as above. An extension of theorem 3 is valid (see Bruno, 1974c, 1975a).

In Bruno [1974c], it is shown that the periodic solutions found by Poincaré [1971], Lyapunov [1935a], Siegel [1956] and others lie in the set $\tilde{\mathscr{A}}$; the conditionally periodic solutions found by Kolmogorov [1954], Arnol'd [1961], Moser [1962, 1967], Bogolyubov [1964], and others lie in the set $\mathscr{B}$.

§ 4. The Normal Form and Methods of Averaging

4.1. Local Coordinates

Let the autonomous system

$$dS/dt \equiv \dot{S} = F(S) \ , \tag{1}$$

where the vector S is $(s_1, \dots, s_{l+m+n})$, be defined and analytic in a domain $\mathscr{G}$ of the complex space $\mathbf{C}^{l+m+n}$ and let it be real in $\operatorname{Re}\mathscr{G}$. In $\operatorname{Re}\mathscr{G}$, let system (1) have an invariant analytic manifold $\mathscr{M}$ of dimension $l + m$, which is fibered into m-dimensional invariant tori $\mathscr{T}^m$. *Local coordinates* for the manifold $\mathscr{M}$ are functions

$$X = (x_1, \dots, x_l) \ , \qquad Y = (y_1, \dots, y_m) \ , \qquad Z = (z_1, \dots, z_n) \tag{2}$$

of S, which are defined and analytic in some complex neighborhood of the manifold $\mathscr{M}$ and possess the following properties: (I) $Z = 0$ on the manifold $\mathscr{M}$; (II) the equations $X = \text{const}$, $Z = 0$ define the m-dimensional invariant torus $\mathscr{T}^m = \{Y \bmod 2\pi\}$; (III) the Jacobian $D(X, Y, Z)/D(S)$ is non-zero on $\mathscr{M}$. The manifold $\operatorname{Re}\mathscr{M}$ is the fiber bundle of the m-dimensional torus $\mathscr{T}^m = \{Y \bmod 2\pi\}$ on the l-dimensional domain $\mathscr{H}$ with respect to X. In local coordinates (2), system (1) has the form

$$\begin{aligned} \dot{X} &= \Phi^{(1)}(X, Y, Z) \ , \\ \dot{Y} &= \Omega(X, Y) + \Phi^{(2)}(X, Y, Z) \ , \\ \dot{Z} &= A(X, Y)Z + \Phi^{(3)}(X, Y, Z) \equiv \underset{\sim}{Z}(X, Y, Z) \ , \end{aligned} \tag{3}$$

where $\Phi^{(k)} = O(|Z|)$, $k = 1, 2$; $\Phi^{(3)} = O(|Z|^2)$; and A is a matrix.

Basic assumption. *There exist local coordinates* (2) *such that in system* (3)

$$\Omega(X, Y) = \Omega = \text{const} \tag{4}$$

and the matrix A is triangular with a constant main diagonal:

$$a_{ij}(X, Y) = 0 \ \textit{for}\ i < j, \ a_{ii}(X, Y) = \lambda_i = \text{const} \ . \tag{5}$$

That is, system (1) is reducible on all invariant tori $\mathscr{T}^m \subset \mathscr{M}$, and it has a unique frequency basis; moreover, the variational system is reducible and its eigenvalues λ_i are the same for every torus $\mathscr{T}^m$. Here, to real coordinates S correspond real coordinates X and Y and complex values of the coordinates Z which are connected by the relation of reality (see section 1.9 of this chapter). We denote by $\Lambda = (\lambda_1, \ldots, \lambda_n)$ the main diagonal of the matrix A.

4.2. The Normal Form

Let the function $f(X; Y)$ be analytic in X and Y and 2π-periodic in Y, where

$$\{X \in \mathscr{H}, |\operatorname{Im} Y| < \varepsilon\} \ . \tag{6}$$

This function can be expanded in a Fourier series

$$f = \sum f_P(X) \exp i\langle P, Y\rangle \ , \qquad P \in \mathbf{Z}^m \ , \qquad (i^2 = -1) \ ,$$

which converges absolutely on the set (6) (recall that $\langle P, Y\rangle = \sum p_i y_i$). All such functions form a ring, which we denote by $\mathscr{P}(X; Y)$. By $\mathscr{P}(X; Y)[[Z]]$ we denote the ring of formal power series

$$f = \sum f_Q(X; Y) Z^Q \ , \qquad 0 \leqslant Q \in \mathbf{Z}^n \ , \tag{7}$$

where $Z^Q = z_1^{q_1} \ldots z_n^{q_n}$ and $f_Q \in \mathscr{P}(X; Y)$. Every series in this ring can be expanded in a unique Taylor-Fourier series (also called a Poisson series)

$$f = \sum f_{PQ}(X) Z^Q \exp i\langle P, Y\rangle \ . \tag{8}$$

Note that the functions $\Phi^{(k)}$ in system (3) allow convergent expansions of the form (7) and (8).

We will now simplify system (3) by means of a formal change of local coordinates

$$\begin{aligned} X &= U + \Xi^{(1)}(U, V, W) \ , \\ Y &= V + \Xi^{(2)}(U, V, W) \ , \\ Z &= W + B(U, V)W + \Xi^{(3)}(U, V, W) \ , \end{aligned} \tag{9}$$

where the series $\xi_j^{(k)} \in \mathscr{P}(U; V)[[W]]$ and contain no terms free of W, the $\xi_j^{(3)}$ contain no linear terms in W, and the matrix B is triangular with zero diagonal. Let transformation (9) carry system (3), (4) (5) into the formal system

$$\begin{aligned} \dot{U} &= \Theta^{(1)}(U, V, W) \ , \\ \dot{V} &= \Omega + \Theta^{(2)}(U, V, W) \ , \\ \dot{W} &= C(U, V)W + \Theta^{(3)}(U, V, W) \equiv \Psi \ , \end{aligned} \tag{10}$$

where the $\theta_j^{(1)}$, $\theta_j^{(2)}$, and ψ_j all belong to $\mathscr{P}(U; V)[[W]]$, the $\Theta^{(k)}$ have no terms

free of W, the $\Theta^{(3)}$ have no linear terms, and C is a triangular matrix with constant diagonal Λ. We write

$$g_j = \psi_j/w_j = \sum g_{jPQ}(U)W^Q \exp i\langle P, V\rangle \;, \qquad j = 1, \ldots, n \;. \tag{11}$$

Here, P runs over $\mathbf{Z}^m$, while Q runs over the set

$$\mathbf{N}_j = \{Q\colon Q \in \mathbf{Z}^n, Q + E_j \geqslant 0, \|Q\| \geqslant 0\} \;;$$

we write $\mathbf{N} = \mathbf{N}_1 \cup \cdots \cup \mathbf{N}_n$ (as we did in sec. 1.2). We will use the usual expansions for $\Theta^{(1)}$ and $\Theta^{(2)}$:

$$\Theta^{(k)} = \sum \Theta^{(k)}_{PQ}(U)W^Q \exp i\langle P, V\rangle \;, \qquad k = 1, 2 \;. \tag{12}$$

We call the formal system (10) a *normal form* if, in expansions (11) and (12), the only non-zero coefficients g_{jPQ} and $\Theta^{(k)}_{PQ}$ are those for which the indices P and Q satisfy the equation

$$i\langle P, \Omega\rangle + \langle Q, \Lambda\rangle = 0 \;. \tag{13}$$

Restriction 1. *For every $Q \in \mathbf{N}$, as $|P| \to \infty$*

$$\underline{\lim}\, |P|^{-1} \ln |i\langle P, \Omega\rangle + \langle Q, \Lambda\rangle| \geqslant 0 \;, \tag{14}$$

where the limit infimum is taken over those $P \in \mathbf{Z}^m$ which do not satisfy equation (13). (*See Bruno*, 1975a.)

Theorem 1. *For the system* (3), (4), (5) *under restriction* 1, *there exists a formal transformation* (9) *to the normal form* (10).

The advantage of the normal form (10) over an arbitrary system (3) lies in the fact that expansions (11) and (12) contain only resonant terms. Hence, the solution of system (10) reduces to integrating a system of lower order (see sec. 2.2). The normal form (10) and normalizing transformation (9) are not uniquely determined by the original system. The normal form (10) for a manifold $\mathcal{M}$ is a generalization of the normal form for a fixed point (§§ 1–3 of this chapter). Properties a)–d) below are the corresponding generalizations:

a) The normalizing transformation preserves the relation of reality between the local coordinates (section 1.9, Chapter III).

b) If one of the equations of the system (3) has the form $\dot{\alpha} = \text{const.}$, then the corresponding coordinate (x_j, or y_j, or z_j) is unchanged under the normalizing transformation. Since a parameter satisfies the equation $\dot{\alpha} = 0$, parameters are unchanged by the normalizing transformation (see section 1.6). Here parameters may appear in any of the coordinates X, Y, or Z. A second consequence is that the time does not change in the normalization of non-autonomous systems since it satisfies the equation $\dot{t} = 1$.

c) As a rule, the normalizing transformation diverges in any neighborhood of the manifold $\mathcal{M}$ (Bruno, 1971, 1972a), but it may converge on some sets defined

with respect to the normal form (§ 3). Let the numbers $\lambda_1, \dots, \lambda_{n'}$ $(n' \leqslant n)$ and $i\omega_1, \dots, i\omega_m$ be commensurable (in particular, $\operatorname{Re}\lambda_1 = \dots = \operatorname{Re}\lambda_{n'} = 0$). The normalizing transformation converges on the set

$$\mathscr{A}(\lambda_1, \dots, \lambda_{n'}) = \{U, V, W\colon \Theta^{(1)} = 0, \Theta^{(2)} = \Omega a,$$
$$\psi_j = \lambda_j w_j a,\ j = 1, \dots, n',\ w_{n'+1} = \dots = w_n = 0\}$$

and this set is analytic for the original system. In this case $\operatorname{Re}\mathscr{A}(\lambda_1, \dots, \lambda_{n'})$ consists of periodic solutions. The construction of an analytic set $\mathscr{B}$ which contains conditionally periodic solutions is generalized analogously.

d) The normal form is useful for the approximate integration of system (1) and the investigation of the stability of the manifold $\mathscr{M}$, especially of its formal stability (see Bruno, 1970a, 1976a; Nekhoroshev, 1971, 1973a, 1973b, 1977).

e) The normalizing transformation commutes with a change of local coordinates on $\mathscr{M}$ of the form $X = X(\tilde{X})$, $Y = C + D\tilde{Y}$, where X is an analytic function with non-zero Jacobian, C is a constant vector, and D is a unimodular matrix.

f) The normalizing transformation likewise commutes with the operation of restriction of the manifold $\mathscr{M}$ to a submanifold $\mathscr{M}'$. That is, the normal form on $\mathscr{M}'$ can be obtained from the normal form on $\mathscr{M}$ by means of a corresponding specification of local coordinates.

4.3. Calculating the Coefficients of the Normal Form

Calculating the coefficients of the normal form (10) may be quite cumbersome. We will see, however, that it may sometimes be simplified. We write out the expansions of the right-hand sides of system (3):

$$\begin{aligned} \Phi^{(k)} &= \sum \Phi^{(k)}_{PQ}(X) Z^Q \exp i\langle P, Y\rangle\ , \qquad k = 1, 2\ , \\ f_j &= \zeta_j/z_j = \sum f_{jPQ}(X) Z^Q \exp i\langle P, Y\rangle\ , \qquad j = 1, \dots, n\ . \end{aligned} \tag{15}$$

Let K be an n-dimensional vector such that, for every non-zero coefficient in the expansions (15), $\langle Q, K\rangle \geqslant 0$ and, additionally, if $\langle Q, K\rangle < s$, then $i\langle P, \Omega\rangle + \langle Q, \Lambda\rangle = 0$, where $s > 0$ is some fixed number. That is, system (3) is *normalized to degree s* (compare with section 1.7).

Theorem 2. *Let a vector K be given, and let system* (3) *be normalized to degree s. Then there exists a normal form of this system in which the coefficients of the terms of degree less than* 2s *are exactly the coefficients of the corresponding resonant terms in system* (3). *That is* $\Theta^{(k)}_{PQ}(U) = \Phi^{(k)}_{PQ}(U)$ and $g_{jPQ}(U) = f_{jPQ}(U)$ *for* $\langle Q, K\rangle < 2s$ *and* $i\langle P, \Omega\rangle + \langle Q, \Lambda\rangle = 0$.

We now note that the resonant part of any of the expressions (15) coincides with the averaging of the corresponding function along solutions of the system $\dot{X} = 0$, $\dot{Y} = \Omega$, $\dot{z}_j = \lambda_j z_j$ $(j = 1, \dots, n)$, if $\operatorname{Re}\lambda_j = 0$ for all j. Consequently, the

concept of "resonant part" is analagous to the concept of "averaging" but is even applicable to situations in which $\operatorname{Re}\Lambda \neq 0$. On the other hand, averaging is defined even for non-analytic functions, for which we cannot write expansions like those in (15), and hence cannot identify resonant parts.

Example 1. Let us consider a system in the *standard form*

$$\dot{X} = \varepsilon F(X,t)\ , \tag{16}$$

where the vector function F is analytic in X and t on some domain $\mathscr{D}$ and is conditionally periodic in t with frequency basis $\Omega = (\omega_1, \ldots, \omega_n)$. Then system (16) is equivalent to the autonomous system

$$\dot{X} = \varepsilon \tilde{F}(X, Y)\ , \qquad \dot{Y} = \Omega\ , \qquad \dot{\varepsilon} = 0\ , \tag{17}$$

where $\tilde{F}(X, \Omega t) = F(X,t)$ and the small parameter ε is the unique small coordinate. By theorem 1 and property b), the normalizing transformation of system (17) has the form

$$X = U + \sum_{k=1}^{\infty} \varepsilon^k H_k(U, V)\ , \qquad Y = V\ , \qquad \varepsilon = \varepsilon\ .$$

Here, the solutions of equation (13) are $P = 0$ and $Q = q_1$, an arbitrary integer. The normal form

$$\dot{U} = \sum_{k=1}^{\infty} \varepsilon^k G_k(U)\ , \qquad \dot{V} = \Omega\ , \qquad \dot{\varepsilon} = 0$$

is "averaged over time". By theorem 2, with $K = 1$ and $s = 1$, the function G_1 is the resonant part of the function $\tilde{F}(U, Y)$ or, equivalently, it is the time-average of the function $F(U,t)$. Thus, the asymptotic method of Krylov-Bogolyubov (see Bogolyubov and Mitropol'skiĭ, 1974) for analytic standard systems is simply a special case of the normal form method. The reduction of the system to standard form is, in fact, just the transformation to local coordinates. In § 2 of chapter V, we will discuss the applications of this method in mechanics.

4.4. A Hamiltonian System

A Hamiltonian system is distinguished by the fact that it is defined by a single Hamiltonian function and by the grouping of all variables (except parameters) into canonical pairs. Local coordinates for the manifold $\mathscr{M}$ also form canonical pairs, so that system (3) is also Hamiltonian (see Nekhoroshev, 1972). The Y coordinates all belong to different pairs. The Hamiltonian function h of system (3) can be expanded in a series

$$h = \sum h_{PQ}(X) Z^Q \exp i\langle P, Y\rangle\ . \tag{18}$$

The normalizing transformation (9) can be taken to be canonical, so the normal form is again a Hamiltonian system; its Hamiltonian is

$$h = \sum g_{PQ}(U)W^{Q}\exp i\langle P, V\rangle \ , \tag{19}$$

which contains only resonant terms (as defined by equation (13)). Properties a)–f) continue to hold here.

Let the Hamiltonian (18) depend on one small parameter μ and have the form $h = h_0 + \mu h_1$, where the expansion of h_0 contains only resonant terms. Then by theorem 2, the Hamiltonian of the normal form (19) may be written as

$$h = h_0 + \mu[h_1] + O(\mu^2) \ , \tag{20}$$

where $[h_1]$ is the resonant part of the "perturbation" h_1.

Example 2. The planar circular restricted three-body problem is a Hamiltonian system with two degrees of freedom and one parameter μ. When $\mu = 0$, the system is integrable; its phase space contains a one-parameter family of circular motions and a continuum of three-dimensional manifolds, corresponding to elliptic motions with a fixed frequency basis (see Bruno, 1972c). For circular motions, $\operatorname{Re}\mathcal{M}_1 = \mathcal{T}^1$ is a cycle (periodic solution), and local coordinates are the second system of Poincaré elements. For elliptic solutions, $\operatorname{Re}\mathcal{M}_2 = \mathcal{T}^2 \times \mathcal{H}$, where $\mathcal{H}$ is the interval $0 < e < 1$; local coordinates are Delaunay elements or the first system of Poincaré elements with constant displacement. In such coordinates, system (3) is a normal form for $\mu = 0$. Thus, for small μ, the Hamiltonian of the normal form is as in of (20). The Delaunay method (in Krasinskiĭ's interpretation, 1973) corresponds to the reduction to normal form in a neighborhood of a cycle $\mathcal{M}_1$ (for circular motion) or of a torus $\mathcal{T}^2$ in the manifold $\mathcal{M}_2$. The method described above allows us to perform the normalizing transformation at once in the neighborhood of the entire manifold $\mathcal{M}_2$ for all values of the eccentricity, $0 < e < 1$.

4.5. On Problems in Mechanics which Can Be Solved with the Aid of the Normal Form

Since, as a rule, the normalizing transformation (9) diverges in any neighborhood of the manifold $\mathcal{M}$ (see Bruno, 1971, 1972a), the correspondence between solutions of the given system (3) and its normal form (10) demands a special study. The results which have been obtained in this direction can be classified as follows.

1) On the stability of the manifold $\mathcal{M}$.

a) The asymptotic stability (or instability) of system (3) can be deduced from the asymptotic stability (or instability) of system (10). Much work has been done here, all of it summarized in hypothesis 2 of Bruno [1970a] (see, for example, Starzhinsky, 1972, 1973, 1974, 1977).

b) The neutral stability of system (10) implies the formal stability of the system (3) (see Nekhoroshev, 1971, 1973a, 1973b, 1977) and in some cases Lyapunov stability (see Moser, 1962, 1968). Results of this type were used, for example, to investigate the stability of the Lagrange solutions of variants of the restricted three-body problem (Markeev, 1973, 1978).

2) Estimating the effects of the instability of system (3) with the help of the system (10). Here, for example, we might calculate the influence of nutational oscillations on the drift speed of a heavy gyroscope in a Cardan suspension (§ 1, ch. V); or one might examine the mechanism of gap formation in the asteroid ring (Bruno, 1970b).

3) Determining periodic and quasi-periodic solutions of the system (3) with the help of the system (10) (see §3 of this chapter). Here, we can find new (asymmetric) solutions to the problem of oscillations of a satellite in the plane of an elliptic orbit, using the asymptotic method of Krylov-Bogolubov according to example 1 (see §2 of ch. V). The computer calculations for example 2 allow us to find all asymmetric periodic solutions of the second kind in the restricted three-body problem (Bruno, 1976c).

The exposition of §4 follows that of Bruno [1976a, 1976d]; for generalizations see Bruno [1979].

Chapter IV
On the Newton Polyhedron

§ 1. A System of Differential Equations

1.1. Introduction

We will examine the system

$$\dot{x}_i = \varphi_i(X) , \qquad i = 1, \ldots, n \tag{1}$$

in a neighborhood of the point $X = (x_1, \ldots, x_n) = 0$, assuming that the functions φ_i are analytic in this neighborhood. Our presentation will closely follow that of the two-dimensional case (Chapter II), but without proofs. The point $X = 0$ is *simple* if at least one of the functions φ_i is non-zero at $X = 0$, and *singular* if $\varphi_i(0) = 0$ for all i. This singular point is called *elementary* if the eigenvalues λ_1, $\ldots$, λ_n of the matrix A of the linear part of system (1) do not all vanish. If all the λ_i are zero, then the origin is a *non-elementary singular point*.

We will investigate the behavior of the integral curves of system (1) in a neighborhood of the origin by introducing new coordinates Y,

$$x_i = \xi_i(Y) , \qquad \xi_i(0) = 0 , \qquad i = 1, \ldots, n , \tag{2}$$

in which system (1) takes the simplest form

$$\dot{y}_i = \psi_i(Y) , \qquad i = 1, \ldots, n . \tag{3}$$

We will assume that these functions ξ_i are at least smooth, and that the Jacobian of transformation (2), $D(\Xi)/D(Y)$, does not vanish at the origin. That is, the change of coordinates (2) is invertible. In simpler cases, such a change of coordinates exists in the entire neighborhood of the origin. In more complicated situations, we will have to divide the neighborhood into several parts, $\mathcal{U}_j^{(d)}(\varepsilon)$, and make an appropriate change of coordinates in each part.

In what follows, our attention will be principally focused on the situation $n = 3$. In those discussions in which n is arbitrary, it would, perhaps, be advisable to assume that $n = 3$ on first reading.

1.2. Simple Points and Elementary Singularities

Theorem 1. *If the origin is a simple point of system* (1), *then there exists a change of variables* (2), *such that, in the new coordinates, system* (1) *takes the form*

$$\dot{y}_1 = 1 \ , \qquad \dot{y}_i = 0 \ , \qquad i = 2, \ldots, n \ .$$

This theorem is similar to theorem 1 in § 1, chapter II, and has an identical proof. It means that exactly one integral curve $\mathscr{F}_0$ passes through the origin, while the remaining integral curves in the neighborhood of the origin are almost parallel to the curve $\mathscr{F}_0$.

If the origin is a singular point of the system (1), then a linear change of coordinates can take the matrix of the linear part into Jordan form. We will assume for system (1) that such a transformation has already been applied, so that

$$\dot{x}_i = \lambda_i x_i + \sigma_i x_{i-1} + \ldots \ , \qquad i = 1, \ldots, n \ ,$$

where $\sigma_i = 0$ if $\lambda_i \neq \lambda_{i-1}$, and $\sigma_i = 0$ or 1 if $\lambda_i = \lambda_{i-1}$. We introduce the vector $\Lambda = (\lambda_1, \ldots, \lambda_n)$, vectors $Q = (q_1, \ldots, q_n)$, and the notation $X^Q = x_1^{q_1} \ldots x_n^{q_n}$. We write system (3) in the form

$$\dot{y}_i = y_i g_i = y_i \sum_{Q \in \mathbf{N}_i} g_{iQ} Y^Q \ , \qquad i = 1, \ldots, n \ . \tag{4}$$

Since the $y_i g_i(Y)$ are series in non-negative integral powers of the variables, then $\mathbf{N}_i$ is a set of integral points $Q = (q_1, \ldots, q_n)$ for which $q_i \geqslant -1$ and $q_j \geqslant 0, j \neq i$. We write $\mathbf{N} = \mathbf{N}_1 \cup \cdots \cup \mathbf{N}_n$.

Theorem 2. *If the origin is an elementary singular point of system* (1), *then there exists a formal change of variables* (2), *which takes system* (1) *into a system* (3) *for which* (*in the notation of* (4)) $g_{iQ} = 0$ *if* $\langle Q, \Lambda \rangle \neq 0$.

That is, system (4) includes only *resonant terms* $y_i g_{iQ} Y^Q$, for which

$$\langle Q, \Lambda \rangle \equiv \sum_{i=1}^{n} q_i \lambda_i = 0 \ . \tag{5}$$

We call such a system (4) a *normal form*. This theorem is analogous to theorem 2 in § 1, Chapter II, and the proof is identical (see § 1, Chapter III).

If the original system is real, then for every complex eigenvalue λ, its complex conjugate $\bar{\lambda}$ also appears among the numbers $\lambda_1, \ldots, \lambda_n$. Let $\lambda_1, \ldots, \lambda_{2l}$ be complex eigenvalues, $\lambda_{l+1} = \bar{\lambda}_1, \ldots, \lambda_{2l} = \bar{\lambda}_l$, and $\lambda_{2l+1}, \ldots, \lambda_n$ be real. Then we can introduce complex-conjugate coordinates $x_{l+k} = \bar{x}_k$ and apply a normalizing transformation which preserves that conjugacy. (I.e., $y_{l+1} = \bar{y}_1, \ldots, y_{2l} = \bar{y}_l$ are complex-conjugate variables, while $y_{2l+1}, \ldots, y_n$ are real; see sect. 1.9 of ch. III).

Let δ be the number of linearly independent integral vectors $Q \in \mathbf{N}$ which satisfy equation (5): we will examine several cases.

1) $\delta = 0$. In this case, the normal form has the form

$$\dot{y}_i = \lambda_i y_i , \qquad i = 1, \ldots, n .$$

Its solutions are $y_i = c_i \exp \lambda_i (t - t_0)$. The normalizing transformation does not always converge in this case (see Bruno, 1971, 1972a), but for real systems there always exists a smooth normalizing transformation (see Sternberg, 1958, 1959). An example of this case is $\lambda_1 = 1$, $\lambda_2 = \sqrt{2}$, $\lambda_3 = -\sqrt{3}$.

2) $\delta = 1$. Let a vector $R \in \mathbf{N}$ satisfy the equation $\langle R, \Lambda \rangle = 0$, and let the greatest common factor of its components $r_1, \ldots, r_n$ be unity (see sect. 2.3 of ch. III).

If $R \geqslant 0$, then solutions $Q \in \mathbf{N}$ of equation (5) are vectors $Q = kR$, where k is a non-negative integer. The normal form therefore has the form

$$\dot{y}_i = \lambda_i y_i + y_i g_i(Y) , \qquad g_i = \sum_{k=1}^{\infty} g_{ik} Y^{kR} , \qquad i = 1, \ldots, n . \tag{6}$$

Here the series g_i depend only on the product Y^R. We shall make this a new independent variable. Letting $S_1 = R$, we choose $n - 1$ integral vectors $S_2, \ldots, S_n$ such that the determinant of the matrix, whose columns are $S_1, \ldots, S_n$, is unity. We then apply the power transformation

$$z_i = Y^{S_i} , \qquad i = 1, \ldots, n . \tag{6'}$$

Then system (6) is transformed into the system

$$(\ln \dot{z}_i) = g_i' \equiv \sum_{k=0}^{\infty} g_{ik}' z_1^k , \qquad i = 1, \ldots, n , \tag{7}$$

where $g_{10}' = \langle R, \Lambda \rangle = 0$. System (7) is easily integrated to

$$\ln z_i = c_i + \int \frac{g_i'(z_1)}{z_1 g_1'(z_1)} dz_1 .$$

As a rule, the normalizing transformation diverges (see section 3.2 of Chapter III and Bruno, 1971, 1972a), but in many cases there exists a smooth normalizing transformation, for example, if $\operatorname{Re} \lambda_i \neq 0$ for all i (see Sternberg, 1958, 1959). However, even when $n = 3$ we find cases when there is no smooth normalizing transformation. An example is $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = -\sqrt{2}$ with $g_1 \equiv 0$ but $g_2/g_3 \not\equiv \lambda_2/\lambda_3$ in the normal form (6).

If one of the coordinates of R is -1 ($r_n = -1$, say), then the normal form has the form

$$\dot{y}_i = \lambda_i y_i , \qquad i = 1, \ldots, n-1 ,$$

$$\dot{y}_n = \lambda_n y_n + a y_1^{r_1} \ldots y_{n-1}^{r_{n-1}} .$$

This system is easily integrable.

3) $\delta = 2$. Let R^1 and R^2 be two integral vectors which form a basis of the set of integral solutions of equation (5); that is, the greatest common divisor of all the minors of the matrix

$$\begin{pmatrix} R^1 \\ R^2 \end{pmatrix} = \begin{pmatrix} r_1^1 & r_2^1 & \dots & r_n^1 \\ r_1^2 & r_2^2 & \dots & r_n^2 \end{pmatrix} \tag{7'}$$

is unity. Then there exist integral vectors $S_3, \dots, S_n$ (see Bruno, 1965, as well as sect. 2.1, ch. III) such that the determinant of the matrix with columns R^1, R^2, S_3, ..., S_n is unity. Then the power transformation (6′), where $S_1 = R^1$ and $S_2 = R^2$, takes the normal form into the system

$$(\ln \dot{z}_i) = g_i'(z_1, z_2) \ , \qquad i = 1, \dots, n \ .$$

A subsystem for the first two variables can be separated from this system

$$(\ln \dot{z}_i) = g_i'(z_1, z_2) \ , \qquad i = 1, 2 \ . \tag{8}$$

If we find $z_1(t)$ and $z_2(t)$, then the remaining variables can be found from

$$\ln z_j = \int g_j'(z_1(t), z_2(t))\, dt \ , \qquad j = 3, \dots, n \ .$$

We must examine subsystem (8) in a neighborhood of the origin $z_1 = z_2 = 0$, which is a non-elementary singular point. We already know how to solve such a problem (see § 3 of ch. II). To do so, we divide the neighborhood of the origin into sets $\mathcal{U}_j^{(d)}(\varepsilon)$, in each of which we then introduce a new set of variables in which system (8) is integrable. In these cases transformation (2) diverges as a rule. We have to seek a smooth transformation to an integrable system in sets corresponding to the sets $\mathcal{U}_j^{(d)}(\varepsilon)$ of system (8), rather than in the entire neighborhood of the origin. But situations may arise in which such a smooth transformation exists only in some of these sets, not in all of them. Then a smooth normalizing transformation does not exist. If $n = 3$, then $\delta = 2$ only if all the λ_i are real and pairwise commensurable, or else $\lambda_1 = 0$ and $\lambda_2 = \overline{\lambda}_3$ are pure imaginary.

4) $\delta \geqslant 3$. Then we can find integral solutions of equation (5) $R^1, \dots, R^\delta$, which can be supplemented with integral vectors $S_{\delta+1}, \dots, S_n$ such that the determinant of the matrix with columns R^1, ..., R^δ, $S_{\delta+1}$, ..., S_n is equal to unity. Then the power transformation (6′), with $S_i = R^i$, $i = 1, \dots, \delta$, takes the normal form (4) into the system

$$(\ln \dot{z}_i) = g_i'(z_1, \dots, z_\delta) \ , \qquad i = 1, \dots, n \ . \tag{8'}$$

(see sect. 2.1 of ch. III). From this system, we can separate out an independent subsystem

$$\dot{z}_i = z_i g_i'(z_1, \dots, z_\delta) \ , \qquad i = 1, \dots, \delta \tag{9}$$

of δ equations. The remaining $n - \delta$ variables can be found from the first δ by

quadrature. We must investigate system (9) in the neighborhood of the point $z_1 = \cdots = z_\delta = 0$, which is a non-elementary singular point. If $\delta > 2$, this is a difficult problem. Thus, we arrive at the problem of examining non-elementary singular points. The transformation to normal form uses the linear terms of the original system to reduce the order of the system from n to $\delta < n$; but if $\delta > 1$, we must study further this system of order δ.

1.3. The Generalized Normal Form

Let $R_1, R_2, \ldots, R_k$ be vectors such that, for some vector T, we have

$$\langle R_i, T\rangle < 0 \ , \qquad i = 1, \ldots, k \ .$$

Then the set of points

$$\mathbf{V} = \{Q\colon Q = \beta_1 R_1 + \cdots + \beta_k R_k, \beta_i \geqslant 0\}$$

is a *polyhedral convex cone.* We will call the series $f = \sum f_Q X^Q$ a series of the class $\mathscr{V}$ if all the exponents Q in the series are integral vectors that belong to the cone $\mathbf{V}$.

Consider the system

$$(\ln\dot{} x_i) = f_i(X) \equiv \sum f_{iQ} X^Q \ , \qquad i = 1, \ldots, n \ , \tag{10}$$

where the f_i are series of class $\mathscr{V}$; let us write $\Lambda = F_0 = (f_{10}, \ldots, f_{n0})$. If the series f_i converge, they do so on a set of the form

$$\mathscr{U}_{\mathbf{V}} = \{X\colon |X|^{R_i} \leqslant \varepsilon, i = 1, \ldots, k\} \ .$$

Theorem 3. *If, in system* (10), *the f_i are series of class $\mathscr{V}$, then there exists a formal change of variables* (2), *where the $y_i^{-1}\xi_i$ are series of class $\mathscr{V}$, which takes system* (1) *into a normal form* (4) *in which the g_i are series of class $\mathscr{V}$ and $g_{iQ} = 0$ if $\langle Q, \Lambda\rangle \neq 0$.*

This is an analog to theorem 1, § 2 in chapter II; their proofs are identical. The set $\mathscr{U}_{\mathbf{V}}$ consists of curvilinear cones which abut the origin.

1.4. The Polyhedron and the Normal Cones

Let $\mathbf{D}$ be a set of points $Q = (q_1, \ldots, q_n)$ in the n-dimensional real space $\mathbf{R}_1^n$, and let $\mathbf{R}_2^n$ be the dual space to $\mathbf{R}_1^n$. We now consider the following problem: for every $P \in \mathbf{R}_2^n$, find a subset $\mathbf{D}_P$ of $\mathbf{D}$, such that

$$\langle Q, P\rangle = r \text{ for } Q \in \mathbf{D}_P \ ,$$

$$\langle Q, P\rangle < r \text{ for } Q \in \mathbf{D}\backslash\mathbf{D}_P \ ,$$

where

$$r = r(P) = \max_{Q \in \mathbf{D}} \langle Q, P \rangle \ .$$

In order to solve this problem, we first note that, for a fixed vector P, the equation

$$\langle Q, P \rangle = c_0 = \text{const} \tag{11}$$

defines a hyperplane $\mathbf{L}$ in $\mathbf{R}_1^n$ which divides that space into two half-spaces: positive $\mathbf{L}^{(+)} = \{\langle Q, P \rangle > c_0\}$ and negative $\mathbf{L}^{(-)} = \{\langle Q, P \rangle \leqslant c_0\}$. The hyperplane (11) is a *supporting hyperplane* for the set $\mathbf{D}$ if its positive half-space has no points of the set $\mathbf{D}$ and if for any hyperplane $\langle P, Q \rangle = c < c_0$, its positive half-space contains points of the set $\mathbf{D}$. We will denote by $\mathbf{L}_P$ the supporting hyperplane which corresponds to the vector P. Its negative half-space $\mathbf{L}_P^{(-)}$ is *the supporting half-space for the set* $\mathbf{D}$. Clearly, $\mathbf{D}_P = \mathbf{L}_P \cap \mathbf{D}$.

In order to describe the sets $\mathbf{D}_P$ for different vectors P, we need to consider the convex hull Δ of the set $\mathbf{D}$:

$$\Delta = \{Q \colon Q = \delta_1 Q_1 + \cdots + \delta_n Q_n, Q_i \in \mathbf{D}, \delta_i \geqslant 0, \textstyle\sum \delta_i = 1\} \ .$$

We denote by Γ the intersection of all the supporting half-spaces $\mathbf{L}_P^{(-)}$ of the set $\mathbf{D}$. It can be shown that the set Γ coincides with the closure of the set Δ (see theorem 1, § 1, ch. I). We call the intersection of the set Γ with the supporting hyperplane $\mathbf{L}_P$ a *face*. The boundary of the closed set Γ consists of faces of different dimensions (a vertex is a face of dimension zero, an edge is a one-dimensional face, etc.). We will denote each of the faces of Γ by $\Gamma_j^{(d)}$, where d is its dimension and j is its number. We write $\mathbf{D}_j^{(d)} = \mathbf{D} \cap \Gamma_j^{(d)}$. If $\Gamma_j^{(d)} = \mathbf{L}_P \cap \Gamma$, then, clearly, $\mathbf{D}_P = \mathbf{D}_j^{(d)}$; that is, all the boundary subsets $\mathbf{D}_P$ are the subsets $\mathbf{D}_j^{(d)}$ which lie on the faces of Γ.

For some fixed $\Gamma_j^{(d)}$, let us consider the set $\mathbf{U}_j^{(d)}$ of vectors $P \in \mathbf{R}_2^n$ for which $\mathbf{L}_P \cap \Gamma = \Gamma_j^{(d)}$. This set

$$\mathbf{U}_j^{(d)} = \left\{P \colon \begin{array}{ll} \langle Q', P \rangle = \langle Q'', P \rangle \ , & Q' \in \Gamma_j^{(d)}, Q'' \in \Gamma_j^{(d)} \ ; \\ \langle Q', P \rangle > \langle Q, P \rangle \ , & Q \in \Gamma \backslash \Gamma_j^{(d)} \end{array}\right\}$$

is a convex cone which we call the *normal cone* of the face $\Gamma_j^{(d)}$.

Let us also consider the cone $\mathbf{V}_j^{(d)}$ which is normal to $\mathbf{U}_j^{(d)}$; that is

$$\mathbf{V}_j^{(d)} = \{Q \colon \langle Q, P \rangle \leqslant 0 \ , \qquad P \in \mathbf{U}_j^{(d)}\} \ .$$

We call this the *tangent cone* to the face $\Gamma_j^{(d)}$.

Under rather weak restrictions on the set $\mathbf{D}$, the set Γ is a *polyhedron* and its boundary $\partial\Gamma$ consists of a finite number of faces $\Gamma_j^{(d)}$. This is the case, for example, if the set $\mathbf{D}$ consists of a finite number of points (see Goldman and Tucker, 1956); there, both $\mathbf{U}_j^{(d)}$ and $\mathbf{V}_j^{(d)}$ are polyhedral cones.

Let the dimension of the polyhedron Γ itself be n. Then the normal cone $\mathbf{U}_j^{(n-1)}$ of the face $\Gamma_j^{(n-1)}$ is a ray orthogonal to $\Gamma_j^{(n-1)}$ and directed out of Γ. The

tangent cone $\mathbf{V}_j^{(n-1)}$ is the half-space bounded by the hyperplane which contains the point $Q = 0$ and is parallel to the face $\Gamma_j^{(n-1)}$. The normal cone $\mathbf{U}_j^{(n-2)}$ is a two-dimensional sector of the plane orthogonal to the face $\Gamma_j^{(n-2)}$. The tangent cone $\mathbf{V}_j^{(n-2)}$ is the intersection of two half-spaces. The normal cone $\mathbf{U}_j^{(1)}$ of the edge $\Gamma_j^{(1)}$ is a part of the hyperplane normal to $\Gamma_j^{(1)}$; the tangent cone $\mathbf{V}_j^{(1)}$ is n-dimensional, and contains the line which is parallel to the edge $\Gamma_j^{(1)}$ and passes through the origin $Q = 0$. The normal cone of a vertex is an n-dimensional polyhedral cone; the tangent cone of a vertex $\Gamma_j^{(0)} = \tilde{Q}$ is likewise a polyhedral convex cone of dimension n. It is the convex hull of the set $\Gamma - \tilde{Q}$ (that is, the polyhedron Γ is parallel transported such that the vertex $\Gamma_j^{(0)}$ becomes the coordinate origin). In general, the tangent cone $\mathbf{V}_j^{(d)}$ is the convex hull of the set $\Gamma - \tilde{Q}$, where $\tilde{Q}$ is any interior point of the face $\Gamma_j^{(d)}$ (that is, the point $\tilde{Q}$ does not lie on a face of dimension less than d).

Example 1. Let $n = 3$ and $\mathbf{D} = \{Q_1, \ldots, Q_6\}$, where $Q_1 = (0, 0, -1)$, $Q_2 = (0, -1, 0)$, $Q_3 = (-1, 0, 0)$, $Q_4 = (0, 0, 0)$, $Q_5 = (1, 0, 0)$, $Q_6 = (0, 1, 0)$. The polyhedron Γ is pictured in figure 82. Its boundary $\partial\Gamma$ consists of five vertices, eight edges, and five faces. The normal cone corresponding to the face $\Gamma_1^{(2)}$ with vertices Q_1, Q_3, Q_6 is

$$\mathbf{U}_1^{(2)} = \left\{P\colon \begin{array}{l} \langle Q_1, P\rangle = \langle Q_3, P\rangle = \langle Q_6, P\rangle \;, \\ \langle Q_1, P\rangle > \langle Q_2, P\rangle \end{array}\right\}$$

or

$$\mathbf{U}_1^{(2)} = \{P\colon -p_3 = -p_1 = p_2 > -p_2\} \;,$$

which is the ray

$$\mathbf{U}_1^{(2)} = c(-1, 1, -1) \;, \qquad c > 0 \;.$$

The tangent cone $\mathbf{V}_1^{(2)}$ is the half-space $\langle Q, P_1\rangle \leqslant 0$, where $P_1 = (-1, 1, -1)$. That is,

$$\mathbf{V}_1^{(2)} = \{Q\colon q_1 - q_2 + q_3 \geqslant 0\} \;.$$

The normal cone of the edge $\Gamma_1^{(1)}$ with vertices Q_1 and Q_6 is

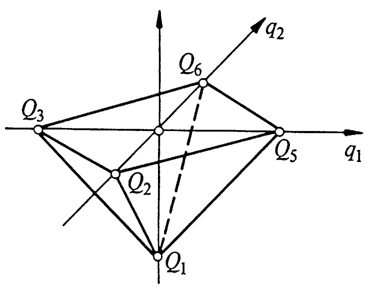

Fig. 82

$$\mathbf{U}_1^{(1)} = \left\{ P: \begin{array}{ll} \langle Q_1, P \rangle = \langle Q_6, P \rangle , & \langle Q_1, P \rangle > \langle Q_3, P \rangle , \\ \langle Q_1, P \rangle > \langle Q_5, P \rangle & \end{array} \right\}$$

or

$$\mathbf{U}_1^{(1)} = \{P: -p_3 = p_2, -p_1 < -p_3, p_1 < -p_3\} .$$

This is a two-dimensional angle which lies in the $p_2 = -p_3$ plane and is bounded by the rays spanned by the vectors P_1 and $P_2 = (-1, -1, 1)$. The tangent cone $\mathbf{V}_1^{(1)}$ is the dihedral angle defined by the inequalities

$$\langle Q, P_1 \rangle \leqslant 0 , \qquad \langle Q, P_2 \rangle \leqslant 0 .$$

That is,

$$\mathbf{V}_1^{(1)} = \{Q: -q_1 + q_2 - q_3 \leqslant 0, -q_1 - q_2 + q_3 \leqslant 0\} .$$

The normal cone corresponding to the vertex $\Gamma_1^{(0)} = Q_1$ is drawn in figure 83. As can be seen, the normal cone of a vertex is bounded by the normal cones of the edges which meet at that vertex. In formal notation,

$$\mathbf{U}_1^{(0)} = \{P: \langle Q_1, P \rangle > \langle Q_j, P \rangle, j = 2, 3, 5, 6\} .$$

Thus, for $n = 2$ or 3, the faces $\Gamma_j^{(d)}$ and their normal cones $\mathbf{U}_j^{(d)}$ can be found from the corresponding drawings. But if $n > 3$, the boundary subsets $\mathbf{D}_j^{(d)}$ and their normal cones can be found from linear relationships among the vectors $Q \in \mathbf{D}$ (see Bruno, 1973b, Soleev, 1983). In particular, the faces $\Gamma_j^{(d)}$ and their normal cones $\mathbf{U}_j^{(d)}$ can be found knowing only the vertices of the polyhedron Γ. Moreover, the normal cone $\mathbf{U}_j^{(d)}$ and the tangent cone $\mathbf{V}_j^{(d)}$ can be described using just the edges which lie on or are adjacent to the face $\Gamma_j^{(d)}$. Let the edges $\Gamma_1^{(1)}, \ldots, \Gamma_l^{(1)}$ lie in a face $\Gamma_j^{(d)}$, and let edges $\Gamma_{l+1}^{(1)}, \ldots, \Gamma_m^{(1)}$ adjoin it. We denote by R_k the unit vector along an edge $\Gamma_k^{(1)}$, so that the vectors $R_{l+1}, \ldots, R_m$ are directed out of $\Gamma_j^{(d)}$ along edges $\Gamma_{l+1}^{(1)}, \ldots, \Gamma_m^{(1)}$, respectively. Then the normal cone is

$$\mathbf{U}_j^{(d)} = \left\{ P: \begin{array}{lll} \langle R_j, P \rangle = 0 , & j = 1, \ldots, l ; \\ \langle R_k, P \rangle < 0 , & k = l + 1, \ldots, m \end{array} \right\} , \tag{12}$$

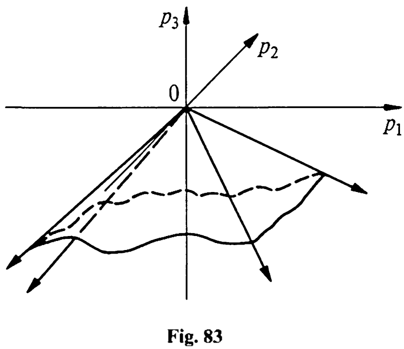

Fig. 83

and the tangent cone is

$$\mathbf{V}_j^{(d)} = \{Q\colon Q = \gamma_1 R_1 + \cdots + \gamma_l R_l + \delta_{l+1} R_{l+1} + \cdots + \delta_m R_m, \delta_k \geqslant 0\} \ .$$

The normal cones of different faces do not intersect, and their union constitutes the normal cone $\mathbf{U}$ of the entire polyhedron Γ. For example, if the set $\mathbf{D}$ lies in the positive orthant $Q > 0$ (i.e., quadrant, octant, and so on), then the cone $\mathbf{U}$ contains the negative orthant $P < 0$.

Thus, in order to identify the boundary subsets $\mathbf{D}_j^{(d)}$ and their normal cones $\mathbf{U}_j^{(d)}$, we must examine the polyhedron Γ, identify its faces $\Gamma_j^{(d)}$, and construct for each of these a normal cone $\mathbf{U}_j^{(d)}$.

For a sum

$$f(X) = \sum f_Q X^Q \tag{12'}$$

the set

$$\mathbf{D} = \mathbf{D}(f) = \{Q\colon f_Q \neq 0\}$$

is called the *support*. The closure of its convex hull is $\Gamma = \Gamma(f)$, the *Newton polyhedron*. To each face $\Gamma_j^{(d)}$ of Γ there corresponds a *truncation*

$$\hat{f}_j^{(d)} = \sum_{Q \in \mathbf{D}_j^{(d)}} f_Q X^Q$$

with respect to the vector order $P \in \mathbf{U}_j^{(d)}$.

We denote by $\mathscr{V}_j^{(d)}$ the class of power series (12′), the supports of which lie in the tangent cone $\mathbf{V}_j^{(d)}$. Together with the normal cone (12), we define for small $\varepsilon > 0$, the set

$$\mathbf{U}_j^{(d)}(\varepsilon) = \left\{P\colon \begin{array}{ll} \ln\varepsilon \leqslant \langle R_j, P\rangle \leqslant -\ln\varepsilon\ , & j = 1, \ldots, l\ ; \\ \langle R_k, P\rangle \leqslant \ln\varepsilon\ , & k = l+1, \ldots, m \end{array}\right\} . \tag{13}$$

If $d = 0$, then the set $\mathbf{U}_j^{(0)}(\varepsilon)$ is the set of values $P = \ln|X|$ for which a series of class $\mathscr{V}_j^{(0)}$ may converge.

1.5. A Non-elementary Singular Point

We consider a system (10) in which the $x_i f_i(X)$ are analytic functions, and the eigenvalues λ_i are all equal to zero. We place each vector $Q \in \mathbf{N}$ in correspondence with a vector coefficient $F_Q = (f_{1Q}, \ldots, f_{nQ})$. We now examine the support $\mathbf{D}$ of the series $\sum F_Q X^Q$, that is, the set $\mathbf{D}(F)$ of vector exponents with non-zero coefficients F_Q:

$$\mathbf{D} = \mathbf{D}(F) = \{Q\colon F_Q \neq 0\} \ .$$

Let $\Gamma = \Gamma(F)$ be the closure of the convex hull of the set $\mathbf{D}$. Generally speaking, Γ is a convex polyhedron (see Goldman and Tucker, 1956, Bruno, 1965, 1973b). It is the *Newton polyhedron of the system* (10). The boundary $\partial\Gamma$ of Γ consists of vertices, edges, faces, etc. We will denote all of these faces of varying dimension by $\Gamma_j^{(d)}$, where d is the dimension and j is the number (see Bruno, 1973b). Thus

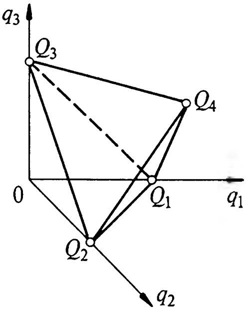

Fig. 84

the $\Gamma_j^{(0)}$ are vertices, the $\Gamma_j^{(1)}$ edges, and so forth. For example, for the system

$$\dot{x}_1 = x_1^2 + 5x_1^2x_2x_3 = x_1(x_1 + 5x_1x_2x_3) \ ,$$

$$\dot{x}_2 = x_2^2 + 4x_1x_2^2x_3 = x_2(x_2 + 4x_1x_2x_3) \ ,$$

$$\dot{x}_3 = x_3^2 + 3x_1x_2x_3^2 = x_3(x_3 + 3x_1x_2x_3) \ ,$$

$$Q_1 = (1,0,0) \ , \qquad Q_2 = (0,1,0) \ , \qquad Q_3 = (0,0,1) \ , \qquad Q_4 = (1,1,1) \ ,$$

Γ is a tetrahedron with four vertices $\Gamma_j^{(0)}$, six edges $\Gamma_j^{(1)}$, and four faces $\Gamma_j^{(2)}$ (figure 84).

For a vector P, the supporting hyperplane $\mathbf{L}_P$ of the set $\mathbf{D}$ is defined by the equation

$$\langle Q, P \rangle = \max_{Q_j \in \mathbf{D}} \langle Q_j, P \rangle \ .$$

The supporting hyperplane $\mathbf{L}_P$ intersects the polyhedron Γ at some face $\Gamma_j^{(d)}$. Since we are interested in the neighborhood of the origin, we need to identify those faces $\Gamma_j^{(d)}$ which lie in the supporting hyperplanes $\mathbf{L}_P$ with $P < 0$. Let this be done. On every edge $\Gamma_j^{(1)}$ we find a unit vector $\pm R_j$, which is uniquely determined (except for sign), as the difference between neighboring integral points on the edge $\Gamma_j^{(1)}$. Now let $\Gamma_j^{(d)}$ be a face of some dimension $0 \leqslant d \leqslant n$. Suppose that the edges $\Gamma_1^{(1)}, \ldots, \Gamma_l^{(1)}$ lie on this face (if $d = 0$, there are none), and let the edges $\Gamma_{l+1}, \ldots, \Gamma_m$ adjoin the face. We choose the vectors $R_1, \ldots, R_l$ with arbitrary direction (sign), but select the vectors $R_{l+1}, \ldots, R_m$ so that they are directed out of the face $\Gamma_j^{(d)}$. We place in correspondence with $\Gamma_j^{(d)}$ a set in X space:

$$\mathcal{U}_j^{(d)}(\varepsilon) = \left\{ X: \begin{array}{ll} \varepsilon \leqslant |X|^{R_i} \leqslant \varepsilon^{-1} \ , & i = 1, \ldots, l \ ; \\ |X|^{R_k} \leqslant \varepsilon \ , & k = l+1, \ldots, m \end{array} \right\} \ .$$

This is the set of all X for which $P = \ln|X|$ belongs to $\mathbf{U}_j^{(d)}(\varepsilon)$ (see expression (13)).

Taken together, all of the sets $\mathscr{U}_j^{(d)}(\varepsilon)$ (for any positive $\varepsilon < 1$) form an entire neighborhood of the origin $X = 0$. In each of these sets $\mathscr{U}_j^{(d)}(\varepsilon)$, we introduce its own set of coordinates in which the system of differential equations becomes simpler. This simplification proceeds differently for different dimensions d of the face $\Gamma_j^{(d)}$.

1) $d = 0$. To investigate system (10) in the set $\mathscr{U}_j^{(0)}(\varepsilon)$, corresponding to the vertex $\Gamma_j^{(0)} = \tilde{Q}$, we make a change of the time variable, $dt_1 = X^{\tilde{Q}}\, dt$ (reduction by $X^{\tilde{Q}}$). Then the vertex $\Gamma_j^{(0)}$ becomes the origin, $F_Q = \Lambda$, and theorem 3 on the generalized normal form applies. That is, system (10) can be transformed, under a change of coordinates of class $\mathscr{V}_j^{(0)}$, to a normal form; this, in turn, can be transformed by a power transformation into a system (8′), where everything reduces to investigating a system of form (9) and of order $\delta < n$. For this system, we must once again examine the neighborhood of a fixed point, but the order of the system has already been reduced. So we must once more construct a polyhedron, divide the neighborhood of the origin into sets $\mathscr{U}_k^{(d)'}(\varepsilon)$, which corresponds to dividing the set $\mathscr{U}_j^{(0)}(\varepsilon)$ into parts $\mathscr{U}_{jk}^{(0)(d)}(\varepsilon)$. In each of the sets $\mathscr{V}_k^{(d)'}(\varepsilon)$ we make a simplification of the system, and we continue the process until we arrive at an integrable system.

2) $d = 1$. To study the behavior of system (10) in a set $\mathscr{U}_j^{(1)}(\varepsilon)$, corresponding to an edge $\Gamma_j^{(1)}$, we must perform a power transformation which turns the edge $\Gamma_j^{(1)}$ into a vertical edge, parallel to the q_n' axis. To do so, we find a set of integral vectors $S_1, \ldots, S_{n-1}$ such that the determinant of the matrix with columns $S_1, \ldots, S_{n-1}, R_j$ is equal to unity (recall that R_j is the unit vector along the edge $\Gamma_j^{(1)}$). We let $y_i = X^{S_i}$, $i = 1, \ldots, n-1$, $y_n = X^{R_j}$. Then the set $\mathscr{U}_j^{(1)}(\varepsilon)$ is transformed into a neighborhood of part of the y_n axis, $\varepsilon \leqslant |y_n| \leqslant \varepsilon^{-1}$. In order to study the system in this neighborhood, we must perform a reduction by the maximal powers of $y_1, \ldots, y_{n-1}$, then find the singular points $y_n^0 \neq 0, \infty$ on the y_n axis and separately study their neighborhoods. We divide the remainder of the neighborhood of the axis into parts, in each of which theorem 1 on simple points applies and the system is integrable. All of this corresponds to dividing the set $\mathscr{U}_j^{(1)}(\varepsilon)$ into parts. In those parts corresponding to simple points on the y_n axis, the system is integrable; in those parts corresponding to singular points y_n^0, we must further investigate the neighborhoods of such points. If the singular point is elementary, then we must put the system into normal form and investigate the transformed, lower-order system. If it is a non-elementary singular point, we must once again construct a polyhedron, divide the neighborhood into parts, and so forth. But this is exactly the problem with which we began, though the singularity is simpler then it was in the original X variables, since it has already been partially resolved.

3) $d = 2$. Let the vectors R^1 and R^2 form an integral basis on the face $\Gamma_j^{(2)}$, i.e., the greatest common divisor of all the minors of matrix (7′) is unity and the plane spanned by R^1 and R^2 is parallel to the face $\Gamma_j^{(2)}$. We supplement the vectors R^1 and R^2 with vectors $S_1, \ldots, S_{n-2}$ to form a basis of the entire space $\mathbf{R}_1^n$; we then perform the power transformation

$$y_i = X^{S_i}\ , \qquad i = 1, \ldots, n-2\ , \qquad y_{n-1} = X^{R^1}\ , \qquad y_n = X^{R^2}\ .$$

Then the face $\Gamma_j^{(2)'}$ is parallel to the q'_{n-1}, q'_n coordinate plane. Now we make a reduction by the maximal powers of $y_1, \ldots, y_{n-2}$. Under our power transformation, the set $\mathcal{U}_j^{(2)}(\varepsilon)$ becomes a set $\mathcal{U}_j^{(2)'}(\varepsilon)$, which is a neighborhood of the part of the y_{n-1}, y_n plane defined by

$$\varepsilon \leqslant |y_{n-1}| \leqslant \varepsilon^{-1}\ , \qquad \varepsilon \leqslant |y_n| \leqslant \varepsilon^{-1}\ .$$

That is, the problem of investigating solutions in the neighborhood of a singular point is reduced to the problem of studying solutions in the neighborhood of part of a plane. As a rule the y_{n-1}, y_n plane is invariant, and we should begin by finding solutions in that plane. We do not know how to completely solve this problem, but we can divide the plane into parts, such that in each part either all points are simple and theorem 1 applies, or there is one fixed point which we can investigate separately. In this case, closed integral curves (for example, limit cycles) will be cut into pieces which will lie in different parts of the y_{n-1}, y_n plane. Division of this plane into pieces corresponds to division of the set $\mathcal{U}_j^{(2)}(\varepsilon)$ into pieces. In pieces corresponding to simple points, the system is integrable; in pieces corresponding to singular points, we must continue our process. Since, in each step, either the singularity is simplified or the dimension is reduced, we must arrive at an integrable system in a finite number of steps.

In principle, the local method allows us to investigate solutions in the neighborhood of periodic solutions. Therefore, in the division of the y_{n-1}, y_n plane we may consider pieces similar to an annulus, a neighborhood of a periodic solution. But three difficulties arise here: finding a periodic solution and normalizing a system in its neighborhood are transcendental (non-algebraic) operations; and an analytic, periodic solution is not, as a rule, algebraic itself. In general, in investigating system (10) in a set $\mathcal{U}_j^{(d)}(\varepsilon)$, we can use a coordinate change (2), where the $y_i^{-1}\xi_i$ are series of class $\mathcal{V}_j^{(d)}$ (compare with section 3.10 of Chapter I and § 4, Chapter II).

4) $d > 2$. Let the vectors $R^1, \ldots, R^d$ form an integral basis of the face $\Gamma_j^{(d)}$, and let the vectors $S_1, \ldots, S_{n-d}$ supplement them to form an integral basis of the entire space. Then the power transformation $y_i = X^{S_i}$, $i = 1, \ldots, n-d$; $y_{n-d+j} = X^{R^j}$, $j = 1, \ldots, d$ transforms $\Gamma_j^{(d)}$ into a face parallel to the $q'_{n-d+1}, \ldots, q'_n$ coordinate subspace. We must then perform a reduction of the system (in Y) by the maximal powers of $y_1, \ldots, y_{n-d}$. The set $\mathcal{U}_j^{(d)}(\varepsilon)$ becomes a neighborhood of part of the $y_{n-d+1}, \ldots, y_n$ coordinate subspace:

$$\varepsilon \leqslant |y_{n-d+j}| \leqslant \varepsilon^{-1}\ , \qquad j = 1, \ldots, d\ .$$

We divide this part of the subspace into pieces, such that in each piece either theorem 1 applies or there is one singular point. We must now investigate the neighborhood of each of the singular points $y_{n-d+1}^0, \ldots, y_n^0$; these singularities are simpler than the original singularity.

Thus, we must resolve singularities, reducing the investigation of solutions in the set $\mathcal{U}_j^{(d)}(\varepsilon)$ (with $d > 0$) to the investigation of the neighborhoods of a few singular points. At elementary singular points, as well as in the sets $\mathcal{U}_j^{(0)}(\varepsilon)$, we can apply theorems 2 and 3 on the normal form. This, together with appropriate power transformations, reduces the order of the system. Thus, our procedure simplifies nonelementary singularities and reduces the order of the system. After a finite number of steps, the neighborhood of the origin $X = 0$ of system (10) is divided into a finite number of sets $\mathcal{U}_{j_1 \ldots j_k}^{(d_1)\ldots(d_k)}(\varepsilon)$, where the indices $j_1, \ldots, j_k$ indicate repeated divisions; in each of these sets new variables are introduced in which the original system is integrable.

1.6. Basic Mathematical Problems

The method, presented in preceding sections, of locally integrating system (1) in the neighborhood of a singular point has three fundamental aspects:

I. The formal-algebraic aspect.

II. The interpretation of formal transformations.

III. The synthesis of the different sets $\mathcal{U}_j^{(d)}(\varepsilon)$ into a general picture.

We now consider these aspects in detail.

I. **The formal-algebraic problem.** We have three kinds of changes of variables:

a) the formal normalizing transformation;

b) the power transformation;

c) the parallel translation.

The question is, what is the simplest system (3) to which we can take an analytic system (1) using these transformations? That is, we seek all such formal systems (3) which cannot be simplified by the transformations mentioned and to which system (1) reduces. The question arises as to the finiteness of the number of simplest systems (3) obtained from one original system. There is also the question of the finiteness of the number of transformations a), b) or c) needed for such a reduction. In addition we must explain whether all of these simplest systems (3) are integrable. In the two-dimensional case ($n = 2$), everything reduced to the integration of a system of order $\delta < n$, i.e., $\delta = 0$ or $\delta = 1$. Such systems are always integrable. These simplest systems are integrable in the general case as well.

II. **Interpretation of formal transformations.** Since normalizing transformations are given as power series which, as a rule, diverge, the question arises of how to interpret these formal transformations. The problem is posed thus: With the help of transformations of types a), b), and c), let the original system (1) in a set $\mathcal{U}^*$ be reduced to the simplest system (3). We seek to determine the properties of solutions of a system (1) in the set $\mathcal{U}^*$ from system (3). There are several possible approaches here—analytic, smooth, topological, formal, and others. That is, we can seek an analytic correspondence between the solutions of the original system

(1) in the set $\mathscr{U}^*$ and the solutions of the simplest system (3). (See § 3, Chapter III). Likewise, we can seek a smooth or continuous transformation from system (1) to system (3). It is also possible to investigate the properties of solutions of system (1) through the properties of the solutions of system (3). For instance, if system (3) has no solutions which approach the singular point, then if there are such solutions of (1), they must approach the singular point very slowly. This leads to the concept of "formal stability" (see sect. 4.5, Ch. III). By studying system (3), it is also possible to describe the properties of solutions of system (1) in a subset of the set $\mathscr{U}^*$ of almost full measure. In my view, the approaches which give the best perspective are the smooth and formal interpretations.

III. **Synthesis.** The method of local integration reduces to the division of the neighborhood of the singularity being investigated into sets

$$\mathscr{U}_{j_1 \dots j_k}^{(d_1) \dots (d_k)}(\varepsilon) , \tag{14}$$

in each of which we introduce new variables for which the system is integrable. It is then necessary to sew together the parts of the solutions found in these pieces of the neighborhood. In some cases, this is not terribly important, but in other cases it is fundamental to the problem. For instance, it is quite important if we wish to determine the stability of a singular point when $n = 2$. If there exist integral curves which approach the origin within one of the pieces of set (14), then the question of sewing together is not important. If there are no such solutions, however, we want to be able to distinguish between a center and a focus and discover the character of a focus. To solve this problem with the local method, it is necessary to divide the neighborhood into sets (14) and to find the solutions in each set to a high degree of accuracy. These solutions define pointwise transformations from one boundary of a curvilinear sector to another. Thus, it is necessary to go through all the sectors of all sets (14) and determine the direction of the displacement of a point after a full rotation around the origin. If (asymptotically as $x_1^2 + x_2^2 \to 0$) this displacement is greater than the error due to approximation, then the origin is a focus and we know its character; if the displacement is less than the error, then we must calculate the solutions with greater accuracy.

Remark. The local method allows us to examine the dependence of solutions of system (1) on some small parameter μ (or several small parameters). We need only add the equation $\dot{\mu} = 0$ to system (1): It is then possible with the local method to study various bifurcations, the creation of periodic solutions, and other local phenomena (see sects. 1.6 and 3.4, Ch. III).

The Newton polyhedron of a Hamiltonian function allows us to find truncated Hamiltonian systems (see Bruno, 1978a).

§ 2. Other Problems Using the Newton Polyhedron

The applications of Newton's polygon to different areas of mathematics have been enumerated in a review by Chebotar'ev [1943]. Since then, the Newton polyhedron has appeared (Bruno, 1962), and a wide range of investigations has been undertaken employing methods of the geometry of exponents. In this part of the chapter, we present a brief survey of those results known to the author (excluding systems of ordinary differential equations, which have considered in § 1 of this chapter).

2.1. The Hadamard Open Polygon

We consider a polynomial in one variable

$$f(z) = \sum_{k=0}^{m} a_k z^k \tag{1}$$

and let $l_k = \ln|a_k|$; then

$$f = \sum \theta_k \exp(l_k + k \ln|z|) \ ,$$

where $|\theta_k| = 1$. In order to select the terms $a_k z^k$ which are largest in modulus, i.e., to find

$$\max_k \, (l_k + k \ln|z|) \ , \tag{2}$$

it is convenient to let $\alpha = \ln|z|$ and employ a method similar to that of the Newton open polygon. We create a correspondence between each term $a_k z^k$ and a point $k = k, l = l_k = \ln|a_k|$ in the k, l plane. Then the maximum (2) is attained on the boundary of the convex hull of all these points. This boundary is an open polygon like the Newton open polygon; it was introduced by Hadamard [1896] to investigate the growth of entire functions (1) with $m = \infty$.

For example, if the number z is a root of the equation $f = 0$, then the number of leading terms $a_k z^k$ must be greater than one (assuming that the sum of the remaining terms is small). That is, the maximum (2) is attained at several points (k, l_k) at once. This is possible only when these points lie on one edge of the

Hadamard open polygon. The slope of this edge determines (approximately) the value of $\alpha = \ln|z|$. See Valiron [1954] as well.

2.2. The Number of Solutions of a System of Algebraic Equations

Let a system of algebraic equations in n complex unknowns, $X = (x_1, \ldots, x_n)$, be given:

$$f_i(X) = 0 \ , \qquad i = 1, \ldots, n \tag{3}$$

Let $\mathbf{D}_i = \mathbf{D}(f_i)$ be the support of the polynomial f_i. We might ask whether we can determine the number of solutions of system (3) from the sets $\mathbf{D}_i$. Bernshtein [1975] found the answer to this question: the number of solutions does not exceed the Minkowski mixed volume of the polyhedrons $\Gamma(f_i)$ (see Busemann, 1958), and is equal to that volume in the general case. From this important theorem follow Kushnirenko's results [Kushnirenko, 1975a, 1975b, Kouchnirenko, 1976] on the number of roots and the multiplicity of critical points of an analytic function (see also Bernshtein, Kushnirenko, and Khovanskiĭ, 1976, Kushnirenko, 1976, Khovanskiĭ, 1978, 1983).

Varchenko [1976] employed the Newton polyhedron to calculate the asymptotics of the integral of a rapidly oscillating function.

2.3. The Separation of Branches of an Algebraic Curve

Let the point $X = 0$ be a singular point of the algebraic curve $\mathscr{F}$ which is defined by the system of equations

$$f_i(X) = 0 \ , \qquad i = 1, \ldots, n-1 \ . \tag{4}$$

That is, $f_i = 0$ and $\partial f_i / \partial x_j = 0$ at $X = 0$. We can find all of the branches of $\mathscr{F}$ which pass through the origin in the following way. For each $i = 1, \ldots, n-1$ we construct a Newton polyhedron $\Gamma_i = \Gamma(f_i)$ and accompanying objects $\Gamma_{ij}^{(d)}$, $\hat{f}_{ij}^{(d)}$, $\mathbf{U}_{ij}^{(d)}$, and $\mathscr{U}_{ij}^{(d)}(\varepsilon)$. We then seek the branches of $\mathscr{F}$ in each of the intersections

$$\mathscr{U}_{1j_1}^{(d_1)}(\varepsilon) \cap \cdots \cap \mathscr{U}_{n-1, j_{n-1}}^{(d_{n-1})}(\varepsilon) \ . \tag{5}$$

After a suitable power transformation

$$x_i = y_1^{\alpha_{i1}} \ldots y_n^{\alpha_{in}} \ , \qquad i = 1, \ldots, n \ . \tag{6}$$

with a unimodular matrix $\alpha = (\alpha_{ij})$, we obtain in the image of set (5) the truncations of the polynomials f_i in the form

$$\hat{f}_{ij_i}^{(d_i)} = y_n^{s_i} h_i(y_1, \ldots, y_{n-1}) \ , \qquad i = 1, \ldots, n-1 \ .$$

System (4), for $y_n \neq 0$, is equivalent to the system

$$f_i y_n^{-s_i} = h_i(y_1, \ldots, y_{n-1}) + \cdots = 0 \ , \qquad i = 1, \ldots, n-1 \ . \tag{7}$$

The desired solutions of this system lie near solutions of the system

$$h_i(y_1, \ldots, y_{n-1}) = 0 \ , \qquad i = 1, \ldots, n-1 \ , \qquad y_n = 0 \ . \tag{8}$$

If a solution $Y = Y^0$ of system (8) is a simple point of system (7), then the corresponding branch of the curve $\mathscr{F}$ can be found by the implicit function theorem. If $Y = Y^0$ is a singular point of system (7), then the translation $Z = Y - Y^0$ will place it at the origin and the procedure of resolving singularities can proceed. After a finite number of steps, as a rule, it is possible to isolate all of the branches of the curve $\mathscr{F}$ (see Bruno, 1972e, 1973a, Soleev, 1983).

For $n > 2$, the determination of the faces of the polyhedron $\Gamma(f)$ and the investigation of their relative disposition is not a simple problem. If the set $\mathbf{D}(f)$ consists of a finite number of points, this can be accomplished by testing all possible combinations. A preprint by the author [Bruno, 1973b] presents a less difficult approach. See also Soleev [1983]. If $n = 3$, it is possible to use graphic methods.

Bugaev [1888] proposed finding the branches of a curve by testing all possibilities; he called it "the principle of greatest and least power exponents". Sintsov [1898] proposed a geometric method for $n = 3$, which can be explained as follows. We consider the normal cones $\mathbf{U}_{ij}^{(d)}$ of the polyhedrons Γ_i in the space $\mathbf{R}_2^n$, dual to $\mathbf{R}_1^n$. Then we consider the cross-section of these cones by some hyperplane (say, $p_1 = -1$). When $n = 3$, this hyperplane is two-dimensional, and the full picture of this cross-section can be represented graphically with straight lines of various colors in a plane. These lines are drawn such that each function f_i has its particular color of lines. Then it is possible to find all the intersections of the cones $\mathbf{U}_{ij}^{(d)}$ and the corresponding truncated functions.

Analogous methods have been applied to the determination of solutions of linear differential equations (see Sintsov, 1898, Chebotar'ev, 1943, Valiron, 1954).

Note that we could solve system (4) by eliminating n-2 coordinates, writing the system as a single equation

$$g(x_1, x_2) = 0 \ ,$$

and solving with Newton's method. But the process of eliminating variables leads to a great increase in the order of the polynomials involved (see Vaĭnberg and Trenogin, 1969). Hence, the methods discussed above require much less computation.

Botashev [1976] suggests a compromise method. Instead of using the Newton polyhedron, he creates a correspondence between each monomial $F_Q x_1^{q_1} \ldots x_{n-1}^{q_{n-1}} x_n^{q_n}$ and a point $p = q_1 + \cdots + q_{n-1}$, $q = q_n$ in the p, q plane. He then constructs a Newton polygon for sets of such points, finds a first approximation to the function, and so forth. The calculations of this method are thus simpler than those of the method of elimination, but are more complicated than the computations of the Newton polyhedron method.

2.4. Linear Partial Differential Equations

Let

$$f(Y) = \sum_{Q \in \mathbf{D}} f_Q Y^Q \tag{9}$$

be some polynomial, let $\mathscr{D} = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)$ be the differential operator, and consider the linear differential equation for a function $u(X)$:

$$f(\mathscr{D})u = 0 \ . \tag{10}$$

In recent years, much work has been done to investigate the properties of solutions of equation (10) using the properties of the Newton polyhedron $\Gamma(f)$ of the symbolic polynomial (9) (see Mikhailov, 1963, Friberg, 1967, Volevich and Gindikin, 1968, 1985, Volevich, 1974, and Gindikin, 1974).

More general equations are also considered. For example, equations with variable coefficients have been examined, and classes of operators with the necessary properties have been distinguished in terms of the Newton polyhedron.

Chapter V
Applications of the Normal Form in Mechanics

§ 1. On the Motion of a Gyroscope in a Cardan Suspension

This part of the chapter is an illustration of the application of the results of § 1, Chapter II and §§ 1, 2, Chapter III. We will calculate the effect of small nutational oscillations on the rate of precession of a gyroscope by reducing the equation of motion to the normal form. We will then show that in the given case the normalizing transformation converges. We will make calculations for a heavy gyroscope with respect to squares and cubes of the amplitude of the nutational oscillations. We will also consider a non-static gyroscope. The mathematical apparatus is presented in sections 1.1–1.4, and its applications to mechanical problems occupy sections 1.5–1.10. These results appeared in an earlier work (Bruno, 1972d). Some of the results were repeated by Zhuravlev [1976]. [Translators note: a Cardan suspension is the same as a suspension in gimbals.]

1.1. Reduction to a Normal Form

We consider a system

$$\dot{x}_i = \varphi_i(X) \ , \qquad \varphi_i(0) = 0 \ , \qquad i = 1, \ldots, n \ , \tag{1}$$

which is analytic at the origin $X = 0$. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of the matrix $(\partial\varphi_i/\partial x_j)_0$. We expand the functions φ_i as power series, and we will seek an invertible change of coordinates

$$x_i = \xi_i(Y) \ , \qquad \xi_i(0) = 0 \ , \qquad i = 1, \ldots, n \ , \tag{2}$$

where the ξ_i are power series, such that in the Y coordinates system (1) will take on the simplest form

$$\dot{y}_i = \psi_i(Y) \ , \qquad i = 1, \ldots, n \ . \tag{3}$$

This is a so-called normal form, in which the series ψ_i contain only resonant terms $cy_1^{p_1} \ldots y_n^{p_n}$ for which

$$\lambda_i = p_1\lambda_1 + p_2\lambda_2 + \cdots + p_n\lambda_n \ .$$

According to theorem 1, § 1, Ch. III, for every system (1) there exists a normalizing

transformation (2), where the ξ_i are formal power series. As a rule, this transformation diverges (Bruno, 1971, 1972a).

Theorem 1. *If, in the analytic system*

$$\begin{aligned} \dot{x}_i &= \varphi_i(x_1, x_2) \ , & i &= 1, 2, 3 \ , \\ \varphi_1(0,0) &= \varphi_2(0,0) = 0 \ , & \lambda_1 &= -\lambda_2 \end{aligned} \tag{4}$$

the subsystem of the first two equations has an analytic integral, then there exists a convergent normalizing transformation

$$\begin{aligned} x_i &= \xi_i(y_1, y_2) \ , & \xi_i(0,0) &= 0 \ , \quad i = 1, 2 \ , \\ x_3 &= y_3 + \zeta(y_1, y_2) \ , & \zeta(0,0) &= 0 \ , \end{aligned} \tag{5}$$

which takes system (4) *into the normal form*

$$\begin{aligned} \dot{y}_1 &= \lambda_1 y_1 + y_1 \sum_{k=1}^{\infty} g_k (y_1 y_2)^k \equiv y_1(\lambda_1 + g) \ , \\ \dot{y}_2 &= -y_2(\lambda_1 + g) \ , \\ \dot{y}_3 &= \sum_{k=0}^{\infty} f_k (y_1 y_2)^k \equiv f \ . \end{aligned} \tag{6}$$

Proof. Since the first two equations form an independent subsystem, we begin by finding the transformations (5.1) and (5.2) which put this subsystem in normal form. The proof that this normal form has the form (6.1), (6.2), and that the normalizing transformation converges, can be found in §§ 14, 15 of Siegel's book [Siegel, 1956] (see also § 1, Chapter II); the existence of the integral is essential to that proof. Having applied the transformation (5.1), (5.2) to system (4), we obtain

$$\dot{x}_3 = \eta(y_1, y_2) \equiv \varphi_3(\xi_1, \xi_2) \tag{7}$$

and we will seek a coordinate change (5.3) which transforms (7) into (6.3). Differentiating formula (5.3) with respect to t and replacing $\dot{x}_3$, $\dot{y}_1$, $\dot{y}_2$, and $\dot{y}_3$ with their expressions from formulae (6) and (7), we obtain an equation for series in y_1 and y_2:

$$\eta = f + \left(\frac{\partial \zeta}{\partial y_1} y_1 - \frac{\partial \zeta}{\partial y_2} y_2\right)(\lambda_1 + g) \ . \tag{8}$$

We decompose the series $\zeta(y_1, y_2)$ into an infinite sum

$$\zeta = \sum_s \zeta^{(s)}(y_1, y_2) \ ,$$

where the $\zeta^{(s)}$ are series containing only those terms $c y_1^{p_1} y_2^{p_2}$ from the expansion

of ζ such that $s = p_1 - p_2$. We similarly obtain $\eta = \sum_s \eta^{(s)}$. Then

$$\frac{\partial \zeta^{(s)}}{\partial y_1} y_1 - \frac{\partial \zeta^{(s)}}{\partial y_2} y_2 = s\zeta^{(s)} .$$

Since the only terms $c y_1^{p_1} y_2^{p_2}$ which appear in the series f are those for which $s = p_1 - p_2 = 0$, then (8) yields

$$\begin{aligned} \eta^{(0)} &= f(y_1 y_2) , \\ \eta^{(s)} &= (\lambda_1 + g)s\zeta^{(s)} , \qquad s \neq 0 . \end{aligned} \tag{9}$$

This in turn leads to

$$\begin{aligned} \zeta^{(s)} &= \frac{\eta^{(s)}}{s(\lambda_1 + g)} , \\ \zeta &= \frac{1}{\lambda_1 + g} \sum_{s \neq 0} \frac{\eta^{(s)}}{s} . \end{aligned} \tag{10}$$

It is evident that the series ζ converges, since the series g and η converge and $|s| > 1$. Formula (9) shows that the series f is simply the sum of the terms $c(y_1 y_2)^k$ from the series expansion of η. The theorem is proved. □

1.2. Integration of the Normal Form

Now suppose that the original system (4) is real (that is, the φ_i are real-valued for real values of the variables), but that the eigenvalues λ_i are pure imaginary. According to section 1.9, chapter II and section 1.9, chapter III, for real values of the X variables the variables y_1 and y_2 will be complex conjugates and y_3 will be real. The solution of system (6) is

$$\begin{aligned} y_1 y_2 &= h = \text{const} , \\ y_1 &= y_1^0 \exp(\lambda_1 t + g(h)t) , \\ y_2 &= y_2^0 \exp(-\lambda_1 t - g(h)t) , \\ y_3 &= y_3^0 + f(h)t , \end{aligned} \tag{11}$$

where

$$f(h) = \sum_{k=0}^{\infty} f_k h^k . \tag{12}$$

For real t, $\operatorname{Re} y_1$ and $\operatorname{Im} y_1$ perform periodic oscillations about the origin, while $y_3(t)$ grows from an initial value with speed $f(h)$ (see (12)). Returning to our original coordinates with our analytic change of coordinates (5), we note that x_1 and x_2 perform periodic oscillations about the origin, while the variable x_3 oscillates and diverges from the origin with a mean speed (12), which depends

on the initial conditions x_1^0 and x_1^0 through the value of the integral h. To find $f(h)$, we must calculate the coefficients f_k and express h in terms of x_1 and x_2. In section 1.4, we will calculate the first coefficient f_1 and express the integral h in terms of x_1 and x_2 to accuracy of order $O\{(x_1^2 + x_2^2)^2\}$.

According to (9), $f_0 = \varphi_3(0,0)$, and the solution with initial conditions $x_1^0 = x_2^0 = 0$ has the form

$$x_1 = x_2 = 0 \ , \qquad x_3 = x_3^0 + f_0 t \ .$$

If $(x_1^0)^2 + (x_2^0)^2 \neq 0$, then

$$x_1 = \beta_1(t) \ , \qquad x_2 = \beta_2(t) \ ,$$
$$x_3 = x_3^0 + f(h)t + \beta_3(t) \ , \tag{13}$$

where the β_i are periodic functions. Thus, the value $f_1 h + f_2 h^2 + \cdots$ gives the deviation of the mean speed of motion of the oscillating coordinate x_3 from the value f_0, which is the speed of motion in the absence of oscillations.

Thus, in agreement with theorem 1, the following holds:

Theorem 2. *Under the conditions of theorem* 1 *and with λ_1 pure imaginary, there exists some $\varepsilon > 0$ such that every solution of system* (4), *with initial conditions such that $|x_1{}^0|^2 + |x_2{}^0|^2 < \varepsilon$, has the form of* (13).

Remark. It is possible to give estimates for ε and for f (as a function of x_1^0, x_2^0) by means of a careful analysis of the convergence proof in Bruno [1971, 1972a, § 5] or in Siegel [1956, § 15].

1.3. Stability with Respect to x_1 and x_2

In what follows, we will treat a particular case of system (4):

$$\dot{x}_1 = x_2 \ ,$$
$$\dot{x}_2 = L(x_1) + M(x_1)x_2^2 \ , \tag{14}$$
$$\dot{x}_3 = N(x_1) \ ,$$

where, for sufficiently small $|x_1|$, the functions L, M, and N have convergent power-series expansions:

$$L = \sum_{k=1}^{\infty} L_k x_1^k \ , \qquad M = \sum_{k=0}^{\infty} M_k x_1^k \ , \qquad N = \sum_{k=0}^{\infty} N_k x_1^k \ . \tag{15}$$

Clearly, the system of the first two equations of (14) is equivalent to a second-order equation

$$\ddot{x}_1 = L(x_1) + \dot{x}_1^2 M(x_1) \ .$$

Since $L_0 = 0$, the point $x_1 = x_2 = 0$ is a rest point for this subsystem of (14).

Theorem 3. *In system* (14), *let* $L_k = 0$ *for* $k < j$ *and* $L_j \neq 0$ *for some* $j, 0 < j < \infty$. *The condition*

$$j - \text{odd} \ , \qquad L_j < 0 \tag{16}$$

is necessary and sufficient for the Lyapunov stability of the solution $x_1 = x_2 = 0$ *with respect to the variables* x_1 *and* x_2.

Proof. System (14) has the analytic integral

$$Kx_2^2 - \int_0^{x_1} 2L(u)K(u)\,du = \text{const} \ , \tag{17}$$

in which

$$K = \exp - \int_0^{x_1} 2M(u)\,du = \exp - 2\sum_{k=0}^{\infty} \frac{M_k x_1^{k+1}}{k+1} \equiv 1 + \cdots$$

By the condition of the theorem

$$\int_0^{x_1} 2L(u)K(u)\,du = \frac{2}{j+1} L_j x_1^{j+1} + \cdots$$

Hence, condition (16) is necessary and sufficient to fix the sign of the integral (17) in some neighborhood of the origin. Further, if integral (17) has fixed sign, then the solution $x_1 = x_2 = 0$ is stable by Dirichlet's theorem (see Siegel, 1956, § 27). If the integral changes sign, then its zeros correspond to those integral curves of the first two equations of (14) which pass through the point $x_1 = x_2 = 0$. Since this is an isolated rest point if $L \not\equiv 0$, and the subsystem of (14) is conservative, then on these integral curves lie both solutions which tend to the point $x_1 = x_2 = 0$ and solutions which leave any arbitrarily small neighborhood of that point. That is, in this case the trivial solution $x_1 = x_2 = 0$ is not stable in the Lyapunov sense. The proof is complete. □

For example, the condition

$$L_1 < 0 \tag{18}$$

is sufficient for stability, while the condition

$$L_1 \leqslant 0 \tag{19}$$

is necessary.

1.4. Calculating the First Terms in $f(h)$

We consider system (14) under condition (18). Since

$$\lambda_{1,2} = \pm\sqrt{L_1}$$

and we have the integral (17), this system satisfies the conditions of theorems 1

and 2. To find the coefficients f_k in (6) and (12), we must calculate the coefficients of transformation (5.1), (5.2), then the coefficients of the series η in (7) and, finally, use equation (9). Here, in order to calculate the coefficient f_k we can limit ourselves to considering terms of degree up to (and including) $2k$ in all our expansions. Let us do this for $k = 1$. We thus write out all the terms through degree 2 in system (14), (15):

$$\begin{aligned} \dot{x}_1 &= x_2 , \\ \dot{x}_2 &= L_1 x_1 + L_2 x_1^2 + M_0 x_2^2 + \cdots \\ \dot{x}_3 &= N_0 + N_1 x_1 + N_2 x_1^2 + \cdots \end{aligned} \tag{20}$$

Here $L_1 < 0$. We let $\lambda = +\sqrt{-L_1}$ (so that $\lambda_1 = i\lambda = -\lambda_2$) and make the linear transformation

$$z_1 = x_1\sqrt{\lambda} , \qquad z_2 = x_2/\sqrt{\lambda} . \tag{21}$$

Then equations (20.1), (20.2) become

$$\begin{aligned} \dot{z}_1 &= \lambda z_2 , \\ \dot{z}_2 &= -\lambda z_1 + \nu z_1^2 + \mu z_2^2 + \cdots , \end{aligned} \tag{22}$$

where

$$\nu = L_2\lambda^{-3/2} , \qquad \mu = M_0\sqrt{\lambda} . \tag{23}$$

In order to put the matrix of the linear part of system (22) into diagonal form, we now move into complex-conjugate coordinates

$$w = z_1 + iz_2 , \qquad \overline{w} = z_1 - iz_2 . \tag{24}$$

Then system (22) becomes

$$\begin{aligned} \dot{w} &= -i\lambda w + \frac{i\nu}{4}(w + \overline{w})^2 - \frac{i\mu}{4}(w - \overline{w})^2 + \cdots \\ \dot{\overline{w}} &= i\lambda\overline{w} - \frac{i\nu}{4}(w + \overline{w})^2 + \frac{i\mu}{4}(w - \overline{w})^2 + \cdots \end{aligned} \tag{25}$$

Next we perform a non-linear change of coordinates

$$\begin{aligned} w &= v + a_{11}v^2 + 2a_{12}v\overline{v} + a_{22}\overline{v}^2 , \\ \overline{w} &= \overline{v} + \overline{a}_{11}\overline{v}^2 + 2\overline{a}_{12}v\overline{v} + \overline{a}_{22}v^2 , \end{aligned} \tag{26}$$

which puts the quadratic terms in (25) into normal form; that is, it completely removes any quadratic terms, which do not appear in the normal form (6.1), (6.2). Thus, with accuracy to third-degree terms,

$$\dot{v} = -i\lambda v , \qquad \dot{\overline{v}} = i\lambda\overline{v} . \tag{27}$$

Differentiating equation (26) with respect to t, we obtain

$$\dot{w} = \dot{v} + 2a_{11}v\dot{v} + 2a_{12}\dot{v}\bar{v} + 2a_{12}v\dot{\bar{v}} + 2a_{22}\bar{v}\dot{\bar{v}}$$

and the conjugate expression for $\dot{\bar{w}}$. We then replace all the derivatives in this equation with their expressions from (25) and (27) and express w in terms of v according to formula (26). We thus obtain an equation which relates two quadratic forms of v and $\bar{v}$. Equating the coefficients of like terms, we obtain linear equations in the unknown a_{ij}. Solving these yields

$$a_{11} = \frac{\mu - \nu}{4\lambda}, \qquad a_{12} = \frac{\nu + \mu}{4\lambda}, \qquad a_{22} = \frac{\nu - \mu}{12\lambda}. \tag{28}$$

Expressing x_1 in terms of v and $\bar{v}$ according to (21), (23), and (26), we find that

$$x_1 = \frac{1}{2\sqrt{\lambda}}\{v + \bar{v} + (a_{11} + \bar{a}_{11})v^2 + 2(a_{12} + \bar{a}_{12})v\bar{v} + (a_{22} + \bar{a}_{22})\bar{v}^2\}.$$

If we substitute this into the series $N(x_1) = N_0 + N_1x_1 + N_2x_1{}^2 + \cdots$ then the coefficient of $v\bar{v}$ is

$$N_1\frac{a_{12} + \bar{a}_{12}}{\sqrt{\lambda}} + N_2\frac{1}{2\lambda}.$$

As we can see from (9), this is just f_1. Thus, using (28), (23) and the equation $L_1 = -\lambda_0^2$, we obtain

$$2\lambda f_1 = N_2 + N_1M_0 - N_1L_2/L_1. \tag{29}$$

Further, inverting transformation (26)

$$v = w - a_{11}w^2 - 2a_{12}w\bar{w} - a_{22}\bar{w}^2 + \cdots$$

and employing formulae (24), (21), (28), and (23), we obtain

$$v\bar{v} = \lambda x_1^2 + \frac{1}{\lambda}x_2^2 - \frac{4M_0\lambda^2 + 2L_2}{3\lambda}x_1^3 - \frac{2M_0}{\lambda}x_1x_2^2 + O\{(x_1^2 + x_2^2)^2\}. \tag{30}$$

Since $h = v\bar{v} + O(|v|^4)$, then the integral h is equal to the right-hand side of (30). Let Δx_1 be the amplitude of the oscillations of x_1, that is $\Delta x_1 = \max|x_1|$. By virtue of the symmetry of the equations this maximum is attained at $x_2 = 0$.

We have thus proved

Theorem 4. *Under condition* (18), *the solutions of system* (14) *have the form of* (13), *and*

$$f(h) = N_0 + \delta_2(\Delta x_1)^2 + \delta_3(\Delta x_1)^3 + O\{(\Delta x_1)^4\}, \tag{31}$$

where

$$\begin{aligned} \delta_2 &= \tfrac{1}{2}(N_2 + N_1M_0 - N_1L_2/L_1), \\ \delta_3 &= \tfrac{2}{3}\delta_2(L_2/L_1 - 2M_0). \end{aligned} \tag{32}$$

Finally, given the system

$$\begin{aligned} \dot{u} &= x_2 , \\ \dot{x}_2 &= l(u) + m x_2^2 , \\ \dot{x}_3 &= n(u) \end{aligned} \tag{33}$$

with l, m, and n analytic functions and $l(u_0) = 0$, then the point $u = u_0$, $x_2 = 0$ is a rest point for the subsystem (33.1), (33.2). If we let $x_1 = u - u_0$, then system (33) takes the form of (20), where

$$L_1 = l'(u_0) , \qquad L_2 = \tfrac{1}{2} l''(u_0) , \qquad M_0 = m(u_0) ,$$

$$N_0 = n(u_0) , \qquad N_1 = n'(u_0) , \qquad N_2 = \tfrac{1}{2} n''(u_0) .$$

Formulas (32) take the form

$$\begin{aligned} \delta_2 &= \tfrac{1}{4}(n'' + 2n'm - n'l''/l') = \tfrac{1}{4} n'(n''/n' + 2m - l''/l') \equiv \tfrac{1}{4} n'\delta . \\ \delta_3 &= \tfrac{2}{3}\delta_2(\tfrac{1}{2} l''/l' - 2m) , \end{aligned} \tag{34}$$

where all terms are evaluated at the point $u = u_0$.

1.5. Statement of the Problem of a Gyroscope in a Cardan Suspension

We consider here the motion of a heavy, symmetric gyroscope in a Cardan suspension, taking into account the mass of the rings of the suspension for the vertical axis of the external ring. We take our notation and equations of motion from a paper by Chetaev [1958]. Let us briefly summarize: ψ is the angle of rotation of the (external) ring (the angle of precession); θ is the angle of tilt of the frame (the angle of nutation); x, y, and z are the axes of the frame; φ is the angle of rotation of the gyroscope in the frame; J is the moment of inertia of the external ring with respect to the vertical axis; A^0, B^0, and C^0 are the moments of inertia of the frame with respect to the x, y, and z axes; A, A, and C are the moments of inertia of a *symmetric* gyroscope with respect to the x, y, and z axes (fig. 85). If friction is absent in the bearings, the only active forces are gravitational, and the center of mass of the frame and the gyroscope lies on the z axis at a distance ζ from the fixed point of the gyroscope, then the equations of motion are

$$\begin{aligned} &(A + A^0)\ddot{\theta} - \dot{\psi}^2(A - C + B^0 - C^0)\cos\theta\sin\theta + C\dot{\varphi}\dot{\psi}\sin\theta = P\zeta\sin\theta , \\ &\frac{d}{dt}\{\dot{\psi}(J + (A + B^0)\sin^2\theta + C^0\cos^2\theta) + C\cos\theta(\dot{\varphi} + \dot{\psi}\cos\theta)\} = 0 , \\ &\frac{d}{dt}(\dot{\varphi} + \dot{\psi}\cos\theta) = 0 . \end{aligned} \tag{35}$$

Here P is the weight of the combined gyroscope and frame. These equations

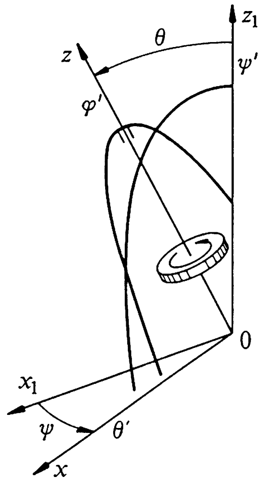

Fig. 85

allow three first integrals of motion, of which we will need only two:

$$\begin{aligned} \dot{\varphi} + \dot{\psi}\cos\theta &= r\ , \\ \dot{\psi}\{J + (A + B^0)\sin^2\theta + C^0\cos^2\theta\} + C\cos\theta(\dot{\varphi} + \dot{\psi}\cos\theta) &= k\ . \end{aligned} \tag{36}$$

Here r and k are suitable constants. Solving equations (36) for $\dot{\varphi}$ and $\dot{\psi}$ and substituting the results into (35.1), we obtain the system

$$\begin{aligned} \ddot{\theta} &= -\frac{b^2}{e}\left\{-\sigma + \frac{\alpha - u}{\gamma - u^2} - u\frac{(\alpha - u)^2}{(\gamma - u^2)^2}\right\}\sin\theta\ , \\ \dot{\psi} &= \frac{b}{e}\frac{\alpha - u}{\gamma - u^2}\ , \\ \dot{\varphi} &= r - u\frac{b}{e}\frac{\alpha - u}{\gamma - u^2}\ . \end{aligned} \tag{37}$$

We have here introduced the notation:

$$\begin{aligned} \sigma &= \frac{P\zeta(A + B^0 - C^0)}{C^2 r^2}\ , \qquad \alpha = \frac{k}{Cr}\ , \\ b &= \frac{Cr}{A + A^0}\ , \qquad e = \frac{A + B^0 - C^0}{A + A^0}\ , \\ \gamma &= \frac{J + A + B^0}{A + B^0 - C^0}\ , \qquad u = \cos\theta \end{aligned} \tag{38}$$

We place the following limits on the parameters:

$$J,\ A^0,\ B^0,\ C^0,\ A,\ C,\ P \geqslant 0\ , \qquad A + A^0 > 0\ ,$$
$$J + A + B^0 > 0\ , \qquad J + C^0 > 0\ , \qquad Cr \neq 0\ .$$

From these inequalities and the relationships (38), we obtain

$$b \neq 0\ , \qquad \gamma < 0 \text{ or } \gamma > 1\ , \qquad \gamma e > 0\ , \qquad |\alpha| < \infty\ .$$

The right-hand side of the first equation of (37) vanishes at stationary values $\theta = \theta_0$. To simplify our calculations, we will consider separately the case when $u_0 = \cos\theta_0$ makes the expression in braces in (37) vanish and the case when $\sin\theta_0 = 0$. A stationary value θ_0 corresponds to regular precession of the gyroscope.

1.6. The General Case $|u_0| < 1$

Since for $|u| < 1$ we have $-\ddot{\theta}\sin\theta = \ddot{u} + \dfrac{u\dot{u}^2}{1-u^2}$, the first equation in (37) can be written

$$\ddot{u} + \frac{u\dot{u}^2}{1-u^2} = \frac{b^2}{e}\left\{-\sigma + \frac{\alpha - u}{\gamma - u^2} - u\frac{(\alpha-u)^2}{(\gamma-u^2)^2}\right\}(1-u^2)\ . \tag{39}$$

If we write $x_3 = \dot{\psi}$, then the system of two equations (39) and (37.2) will be equivalent to system (33), with

$$\begin{aligned} l(u) &= \frac{b^2}{e}\left\{-\sigma + \frac{\alpha - u}{\gamma - u^2} - u\frac{(\alpha-u)^2}{(\gamma-u^2)^2}\right\}(1-u^2)\ ,\\ m(u) &= -\frac{u}{1-u^2}\ , \qquad n(u) = \frac{b}{e}\frac{\alpha - u}{\gamma - u^2}\ . \end{aligned} \tag{40}$$

For ease of calculation, we introduce auxiliary functions

$$\begin{aligned} G(u) &= \frac{\alpha - u}{\gamma - u^2}\ ,\\ F(u) &= -\sigma + G - uG^2\ ,\\ P_2(u) &= u^2 - 2\alpha u + \gamma\ ,\\ P_3(u) &= (\alpha - u)^2(\gamma - u^2) + P_2^2\\ &= -2\alpha u^3 + 3(\alpha^2 + \gamma)u^2 - 6\alpha\gamma u + \alpha^2\gamma + \gamma^2\ . \end{aligned} \tag{41}$$

From (41) we obtain

$$G' = -\frac{1 - 2uG}{\gamma - u^2} = -\frac{P_2}{(\gamma - u^2)^2} ,$$
$$F' = -G^2 - \frac{(1 - 2uG)^2}{\gamma - u^2} = \frac{P_3 G'}{(\gamma - u^2)P_2} = -\frac{P_3}{(\gamma - u^2)^3} . \tag{42}$$

Now let $u = u_0$ be a root of the equation $l(u) = 0$, with $|u_0| < 1$. Since $l = b^2 e^{-1}(1 - u^2)F$, then u_0 must be a root of the equation $F(u) = 0$, or

$$Q(u) \equiv \sigma(\gamma - u^2)^2 + (\alpha - u)(\alpha u - \gamma) = 0 . \tag{43}$$

Thus, when $u = u_0$, we have

$$l' = b^2 e^{-1}(1 - u^2)F' ,$$
$$l'' = -4b^2 e^{-1} uF' + b^2 e^{-1}(1 - u^2)F'' , \tag{44}$$
$$l''/l' = -\frac{4u}{1 - u^2} + F''/F' .$$

Theorem 6. *If* $|u_0| < 1$, *then the solution* $u = u_0$, $\dot{u} = 0$ *of system* (37) *is stable with respect to the variables* $\dot{\theta}$, θ, *and* $\dot{\psi}$ *if and only if one of the two conditions*

$$A + B^0 - C^0 \geqslant 0 \tag{45}$$

or

$$A + B^0 - C^0 < 0 \text{ and } P_3(u_0) > 0 . \tag{46}$$

is satisfied. In the stable case, the mean rate of change of the angle ψ *is*

$$[\dot{\psi}] = n(u_0) + \delta_2(\Delta u)^2 + \delta_3(\Delta u)^3 + O\{(\Delta u)^4\} , \tag{47}$$

where δ_2 *and* δ_3 *are given by formulas* (54).

Proof. By theorem 3, we have stability with respect to the variables mentioned when

$$l'(u_0) < 0 , \tag{48}$$

and instability if either

$$l'(u_0) > 0 \tag{49}$$

or

$$l'(u_0) = 0 \text{ and } l''(u_0) \neq 0 . \tag{50}$$

Let $A + B^0 - C^0 > 0$; then $e > 0$, $\gamma > 1$, and $F' \leqslant 0$, as we can see from the first expression for F' in (42). The equation $F' = 0$ is attained only when $u_0 = \alpha$ and $\gamma = \alpha^2$, which is impossible when $\gamma > 1 \geqslant |u_0|$. Since $1 - u^2 > 0$, then (44.1) implies that (48) is satisfied. Stability in this case was proved by Rumyantsev [1958].

The case $A + B^0 - C^0 = 0$ will be treated in sections 7, 8.

Now let $A + B^0 - C^0 < 0$; then $e < 0$, $\gamma < 0$, and, by (42) and (44),

$$l' = -b^2 e^{-1}(1 - u^2)P_3(\gamma - u^2)^{-3} \ .$$

Therefore (48) is satisfied for $P_3 > 0$, while if $P_3 < 0$, the instability condition (49) is fulfilled. Finally, if $P_3 = 0$, then $l'(u_0) = 0$; however, as we shall now show, $l''(u_0) \neq 0$ and, by (50), we will have instability. In fact, by (44)

$$l'' = b^2 e^{-1}(1 - u^2)F'' = -b^2 e^{-1}(1 - u^2)P_3'(\gamma - u^2)^{-3} \ .$$

It remains to be shown that the system of three equations

$$Q = 0 \ , \qquad P_3 = 0 \ , \qquad P_3' = 0$$

does not have a common solution. Since $P_3' = 6(\alpha - u)(\alpha u - \gamma)$, then the equation $P_3' = 0$ has two roots, $u_0 = \alpha$ and $u_0 = \gamma/\alpha$. By (41)

$$P_3(\alpha) = (\gamma - \alpha^2)^2 > 0 \ ,$$

$$P_3(\gamma/\alpha) = \gamma^3/\alpha^2 + \alpha^2\gamma - 2\gamma^2 < 0 \ ,$$

since $\gamma < 0 < \alpha^2$.

Now let one of the stability conditions (45) or (46) be satisfied; then the condition (18) is satisfied and, by theorem 4, we can calculate the coefficients δ_2 and δ_3 in (47). By (42), (44), and (41), with $u = u_0$, we have

$$\begin{aligned} \delta = n''/n' + 2m - l''/l' &= (\ln G')' - \frac{2u}{1 - u^2} \\ + \frac{4u}{1 - u^2} - (\ln F')' &= \frac{2u}{1 - u^2} - \left\{ \ln \frac{P_3}{(\gamma - u^2)P_2} \right\}' \\ &= \frac{2u}{1 - u^2} - \frac{2u}{\gamma - u^2} + \frac{P_2'}{P_2} - \frac{P_3'}{P_3} \\ &= \frac{2u(\gamma - 1)}{(1 - u^2)(\gamma - u^2)} + \frac{P_2'}{P_2} - \frac{P_3'}{P_3} = \frac{S}{(1 - u^2)(\gamma - u^2)P_2P_3} \ , \end{aligned} \tag{51}$$

where

$$P_2' = -2(\alpha - u) \ , \qquad P_3' = 6(\alpha - u)(\alpha u - \gamma) \ ,$$

$$\begin{aligned} S &= 2(\gamma - 1)uP_2P_3 + (1 - u^2)(\gamma - u^2)(P_2'P_3 - P_3'P_2) \\ &= 2(\gamma - 1)uP_2P_3 - 2(1 - u^2)(\gamma - u^2)(\alpha u^3 - 3\alpha^2 u^2 + 3\alpha\gamma u + \alpha^2\gamma - 2\gamma^2) \ . \end{aligned} \tag{52}$$

Similarly, we find

$$n' = -\frac{b}{e}\frac{P_2}{(\gamma - u^2)^2}\,, \qquad (53)$$

$$\delta_1 = \frac{1}{2}l''/l' - 2m = \frac{1}{2}(\ln F')' = \frac{3u}{\gamma - u^2} + \frac{P_3'}{2P_3}\,.$$

According to (34), $f(h)$ is expressed by formula (31), where

$$\delta_2 = \frac{1}{4}n'\delta = -\frac{1}{4}\frac{b}{e}\frac{S}{(\gamma - u^2)^3(1 - u^2)P_3}\,, \qquad (54)$$

$$\delta_3 = \frac{2}{3}\delta_2\delta_1 = \frac{1}{3}\delta_2\left(\frac{6u}{\gamma - u^2} + \frac{P_3'}{P_3}\right)\,.$$

The theorem is proved. □

The coefficient $f_1 = 0$, i.e., $f(h) = O\{(\Delta u)^4\}$, only if $\delta = 0$, i.e. $S = 0$. The two algebraic equations $Q = 0$ and $S = 0$ relate u_0 and the three parameters σ, α, and γ. If these four variables satisfy both equations, then a solution of system (37) with initial conditions

$$u^0 = u_0\,, \qquad \dot{u}^0 = 0\,, \qquad \dot{\psi}^0 = n(u^0)\,, \qquad \dot{\varphi}^0 = r - u^0 n(u^0)$$

corresponds to the regular precession of the gyroscope. With initial conditions close to these, the gyroscope will undergo pseudo-regular precession. Moreover, the speed of this pseudo-regular precession will be fixed to an accuracy of $O\{(\Delta u)^4\}$. It might be interesting to investigate these dependencies in greater detail.

Remark. Since system (37) is integrable, then its solutions could be studied by integrating, as was done in a paper by Lunts and Smolitsky [1966] and a book by Lunts [1976]. As Smolitsky noted, the results of these works yield condition (45), with the strict inequality, and the first two terms in expression (47).

1.7. The General Case $|u_0| = 1$

Here $\theta_0 = 0$ or π, $u_0 = \cos\theta_0 = \pm 1$, and we go from system (37) to system (33), choosing

$$x_1 = \theta - \theta_0\,, \qquad x_2 = \dot{\theta}\,, \qquad x_3 = \psi\,.$$

Then

$$l(\theta) = -b^2 e^{-1} F(\cos\theta)\sin\theta\,, \qquad m \equiv 0\,, \qquad (55)$$

$$n = \frac{b}{e}\frac{\alpha - \cos\theta}{\gamma - \cos^2\theta}\,.$$

Theorem 7. *If* $|u_0| = 1$, *then the solution* $\theta = \theta_0$, $\dot{\theta} = 0$ *of system* (37) *is stable with respect to the variables* θ, $\dot{\theta}$, *and* $\dot{\psi}$ *if and only if one of the following three conditions is satisfied*:

$$\begin{aligned} &1)\ -e^{-1}F(u_0)u_0 < 0\ ; \\ &2)\ F(u_0) = 0,\ e^{-1}F'(u_0) < 0\ ; \\ &3)\ F(u_0) = 0,\ F'(u_0) = 0,\ -e^{-1}F''(u_0)u_0 < 0\ . \end{aligned} \tag{56}$$

When the first of these conditions is fulfilled, the mean speed of precession is

$$\begin{aligned} [\dot{\psi}] = {} & \frac{k - Cru_0}{J + C} \\ & + \frac{(J + 2A - 2B^0 + C^0)Cr - 2k(A + B^0 - C^0)u_0}{(J + C)^2} u_0(\Delta\theta)^2 + O\{(\Delta\theta)^4\}\ . \end{aligned} \tag{57}$$

Proof. Since the expansion of the function l in powers of $x_1 = \theta - \theta_0$ contains only odd powers of x_1, then, in accordance with theorem 3, any one of these three conditions is sufficient for stability:

$$\begin{aligned} &dl/d\theta < 0\ , && \\ &dl/d\theta = 0\ , && d^3l/d\theta^3 < 0\ , \\ &dl/d\theta = d^3l/d\theta^3 = 0\ , && d^5l/d\theta^5 < 0\ , \end{aligned} \tag{58}$$

where the derivatives are evaluated at $\theta = \theta_0$. At the same time, any of the following three conditions is sufficient to establish instability:

$$\begin{aligned} &dl/d\theta > 0\ ; && \\ &dl/d\theta = 0\ , && d^3l/d\theta^3 > 0\ , \\ &dl/d\theta = d^3l/d\theta^3 = 0\ , && d^5l/d\theta^5 > 0\ . \end{aligned} \tag{59}$$

By (55)

$$\begin{aligned} dl/d\theta &= -b^2e^{-1}F(u_0)u_0\ , \\ d^3l/d\theta^3 &= b^2e^{-1}\{F(u_0)u_0 + 3F'(u_0)u_0^2\}\ , \\ d^5l/d\theta^5 &= -b^2e^{-1}\{F(u_0)u_0 + 15F'(u_0)u_0^2 + 15F''(u_0)u_0^3\}\ . \end{aligned} \tag{60}$$

Therefore conditions (56) coincide with conditions (58), and, consequently, they are sufficient to show stability. The case $F(u_0) = F'(u_0) = F''(u_0) = 0$ is impossible, as shown in the proof of the preceding theorem. Hence, if none of the conditions (56) is fulfilled, then one of the conditions (59) must be satisfied, implying instability.

In what follows, we will assume that the first stability condition (56) is satisfied, and then apply theorem 4. Since, when $\theta = \theta_0$, we have

$$\frac{d^2 l}{d\theta^2} = 0 \ , \qquad \frac{d^2 n}{d\theta^2} = \frac{b(1 - 2\alpha u_0 + \gamma)u_0}{e(\gamma - 1)^2} \ ,$$

then by (31) and (34)

$$f(h) = n(\theta_0) + \frac{1}{4}\frac{d^2 n}{d\theta^2}(\Delta\theta)^2 + O\{(\Delta\theta)^4\} \ ,$$

which is equivalent to formula (57). The proof is complete. □

Remark. In contrast to theorem 6, theorem 7 tells us how to calculate the precession rate in only one of the cases of stability. This is related to the fact that in cases (56.2) and (56.3), $\lambda_1 = \lambda_2 = 0$ and theorem 4 is not applicable. It is, however, still possible to find the precession mode. To do so, we must consider system (14), (15) with $M \equiv 0$, where N is an even, and L an odd, function of x_1, and, in addition, either $L_1 = 0$, $L_3 < 0$ or $L_1 = L_3 = 0$, $L_5 < 0$. It is convenient to use the functions $C_n(\theta)$ and $S_n(\theta)$ introduced by Lyapunov [1935b] (see also §4, Chapter II).

Condition (56.1) is equivalent, when $u_0 = 1$, to condition (2.8) in Rumyantsev [1958]; it is shown there that that condition is sufficient, and the condition $-e^{-1}F(u_0)u_0 \leqslant 0$ is necessary, for the stability of that solution with respect to $\dot\theta$, θ, and $\dot\psi$. The case $F(u_0) = 0$ was not treated there.

It is not hard to derive from (42) the fact that condition (56.2) is equivalent to the condition $Q(u_0) = 0$ and either $A + B^0 - C^0 \geqslant 0$, or $A + B^0 - C^0 < 0$ with $P_3(u_0) > 0$. Condition (56.3) is equivalent to the condition

$$Q(u_0) = 0 \ , \qquad A + B^0 - C^0 < 0 \ , \qquad P_3(u_0) = 0 \ , \qquad \zeta u_0 < 0 \ .$$

As is evident from (57), the speed of revolution of the external ring is, to an accuracy of $O\{(\Delta\theta)^4\}$, independent of the amplitude of the nutational oscillation $\Delta\theta$, if

$$2u_0\frac{k}{r} = \frac{J + 2A + 2B^0 - C^0}{A + B^0 - C^0}C \ .$$

1.8. The Case $A + B^0 - C^0 = 0$

Here, in accordance with (38)

$$e = 0 \ , \qquad e^{-1}\sigma = \frac{P\zeta(A + A^0)}{C^2 r^2} \ , \qquad e\gamma = \frac{J + A + B^0}{A + A^0} > 0 \ ,$$

$$\frac{F(u)}{e} = -\frac{\sigma}{e} + \frac{\alpha - u}{e\gamma} \ . \tag{61}$$

If we restrict ourselves to cases where $|u_0| < 1$, then the equation for u_0 is $e^{-1}F(u_0) = 0$, whence

$$u_0 = \alpha - \sigma\gamma = \frac{k}{Cr} - \frac{P\zeta(J + A + B^0)}{C^2 r^2} . \tag{62}$$

Since, by (61),

$$e^{-1}F'(u) = -(e\gamma)^{-1} < 0 ,$$

we will always have stability here. Further, in formulas (51) and (53) we have rational functions of γ. Letting γ approach infinity and taking into account the limiting equalities (61), we find, in the limit, that

$$\delta = \frac{2u}{1 - u^2} , \qquad n' = -b(e\gamma)^{-1} , \qquad \delta_1 = 0 .$$

From these, using (54), we obtain $\delta_3 = 0$ and

$$\delta_2 = \frac{1}{4}n'\delta = -\frac{bu_0}{2e\gamma(1 - u_0^2)} = -\frac{Cru_0}{2(J + A + B^0)(1 - u_0^2)} .$$

Since (37.2) has, in the given case, the form

$$\dot{\psi} = \frac{k - Cru}{J + A + B^0} \equiv n(u) ,$$

we have

$$\Delta\dot{\psi} = -\frac{Cr}{J + A + B^0}\Delta u ,$$

and formula (47) takes the form

$$[\dot{\psi}] = n(u_0) - \frac{Cru_0(\Delta u)^2}{2(J + A + B^0)(1 - u_0^2)} + O\{(\Delta u)^4\}$$

$$= n(u_0) - \frac{(J + A + B^0)u_0}{2Cr(1 - u_0^2)}(\Delta\dot{\psi})^2 + O\{(\Delta\dot{\psi})^4\} .$$

If we disregard an estimate of the remainder terms $O\{\dots\}$ in this last expression, then we obtain formula (6.6) from a paper by Klimov and Stepanenko [1967] which treats the case $A + B^0 - C^0 = 0$. In particular, $\delta_2 = 0$ only if $u_0 = 0$, that is, according to (62) if $Crk = P\zeta(J + A + B^0)$.

1.9. The Balanced Gyroscope

For a balanced gyroscope, $\zeta = 0$, so that $\sigma = 0$. Equation (43) takes the form $(\alpha - u)(\alpha u - \gamma) = 0$ and has two roots:

$$u = u_0 = \alpha \tag{63}$$

and

$$u = u_0 = \gamma/\alpha . \tag{64}$$

First we consider case (63). Since $n(\alpha) = 0$, this corresponds to a rotation of the gyroscope with fixed axis. As we see from (44) and (42), this always means $l' < 0$. By (52), $P_2' = P_3' = 0$ and formulas (51) and (53) give us

$$\delta = \frac{2u(\gamma - 1)}{(1 - u^2)(\gamma - u^2)}, \qquad \delta_1 = \frac{3u}{\gamma - u^2}.$$

Since $P_2(\alpha) = \gamma - \alpha^2$, then $G'(\alpha) = -(\gamma - \alpha^2)^{-1}$, whence with formula (54) we obtain

$$\delta_2 = \frac{1}{4} n'\delta = -\frac{b}{2e}\frac{\alpha(\gamma - 1)}{(1 - \alpha^2)(\gamma - \alpha^2)^2}$$

$$= -\frac{(J + C^0)Cr\cos\theta_0}{2\{(J + A + B^0)\sin^2\theta_0 + (J + C^0)\cos^2\theta_0\}^2 \sin^2\theta_0},$$

$$\delta_3 = \frac{2}{3}\delta_1\delta_2 = -\frac{b}{e}\frac{\alpha^2(\gamma - 1)}{(1 - \alpha^2)(\gamma - \alpha^2)^3}.$$

Since $\Delta u \approx -\sin\theta_0 \Delta\theta$ then our expression for $\delta_2(\Delta u)^2$ agrees to within $O\{(\Delta\theta)^3\}$ with formula (20) in Magnus [1955], where the rate of drift of a gyroscope axis was calculated for the first time, but with a completely different method and with no proof of convergence. This was followed by numerous works in which the rate of the drift was found to degree $O\{(\Delta\theta)^4\}$ (see Appendix II of the book by Nikolai, 1964).

Smolitsky [1966] calculated the coefficients s_1 and s_2 in the expansion

$$[\dot\psi] = \sum_{k=1}^{\infty} s_k \kappa^k$$

where the parameter κ equals $\gamma(\Delta u)^2\{\gamma - (\alpha + \Delta u)^2\}^{-1}$. Here, κ plays the same role as did the integral h in (11). If we rewrite this series as a series in powers of Δu, then the formulas of Smolitsky agree with our formulae up to order $O\{(\Delta u)^4\}$; however, Smolitsky's calculations give $[\dot\psi]$ to greater accuracy.

We also mention here the paper of Kobrin and Martynenko [1971], which examines a rapidly rotating unbalanced gyroscope. There, the value $[\dot\psi]$ is calculated to within $O(\mu^3)$, where $\mu = \max[\sigma, \Delta u]$, for small σ and $u_0 - \alpha$. An application of the asymptotic method in the paper is not necessary for the case as the result is obtained from Magnus's formula with the following simple calculations. In the neighborhood of $u_0 = \alpha$, $\sigma = 0$, formula (47) yields

$$[\dot\psi] = n'(\alpha)(u_0 - \alpha) + \tfrac{1}{2}n''(\alpha)(u_0 - \alpha)^2 + \delta_{20}(\Delta u)^2 + O(\mu^3). \qquad (65)$$

Here $\delta_{20} = \delta_2$ at $\sigma = 0$, $u_0 = \alpha$; that is, $\delta_{20}(\Delta u)^2$ is Magnus's drift rate of the balanced gyroscope. In a neighborhood of the point $u_0 = \alpha$, $\sigma = 0$, the root of equation (43) can be expanded into a power series in σ:

$$u_0 - \alpha = (\alpha^2 - \gamma)\sigma + 3\alpha(\alpha^2 - \gamma)\sigma^2 + O(\sigma^3).$$

Substituting this into (65) and considering only those terms of no more than second degree in Δu and σ, we obtain formula (3.4) from the paper by Kobrin and Martynenko [1971].

We next consider case (64). Here we have a regular precession with rate $n(\gamma/\alpha) \neq 0$, which is of the same order of magnitude as the speed of the proper rotation of the gyroscope. According to (52), $P_3' = 0$, $P_2'/P_2 = 2\alpha/\gamma$, and formulas (51) and (53) give

$$\delta = \frac{2u(\gamma - 1)}{(1 - u^2)(\gamma - u^2)} + \frac{P_2'}{P_2} = \frac{2\alpha(\alpha^4 - 2\alpha^2\gamma + \gamma^3)}{\gamma(\alpha^2 - \gamma^2)(\alpha^2 - \gamma)} ,$$

$$\delta_1 = \frac{3u}{\gamma - u^2} = \frac{3\alpha}{\alpha^2 - \gamma} , \qquad n' = -\frac{b}{e}\frac{\alpha^2}{\gamma(\alpha^2 - \gamma)} .$$

Therefore

$$\delta_2 = \frac{1}{4}n'\delta = -\frac{b}{2e}\frac{\alpha^3(\alpha^4 - 2\alpha^2\gamma + \gamma^3)}{\gamma^2(\alpha^2 - \gamma)^2(\alpha^2 - \gamma^2)} .$$

Since $|\gamma/\alpha| < 1$ and $\gamma(\gamma - 1) > 0$, then we must have $\alpha^4 - 2\alpha^2\gamma + \gamma^3 > 0$; hence δ_2 never vanishes.

Finally, the analysis of the case $|u_0| = 1$ follows immediately from section 1.7.

1.10. The Spherical Pendulum

The spherical pendulum is a point of mass m which moves on a sphere of radius ζ (see Sommerfeld, 1947, § 18). It can be considered as a special case of our Cardan gyroscope with the following values of the parameters:

$$J = A^0 = B^0 = C^0 = 0 , \qquad A = \zeta^2 m , \qquad P = mg ; \tag{66}$$

$$Cr = 0 . \tag{67}$$

In order to apply theorem 6, we let the values of the parameters in (66) be given, but in place of (67), we take $Cr = \varepsilon$ and let ε approach zero. Then

$$\sigma = \frac{P\zeta A}{C^2r^2} = \frac{m^2g\zeta^3}{\varepsilon^2} , \qquad \alpha = \frac{k}{\varepsilon} , \qquad b = \frac{\varepsilon}{A} , \qquad e = \gamma = 1 .$$

Since theorem 6 remains valid for $\gamma = 1$, and $A + B^0 - C^0 > 0$, then the basic motion $u = u_0$ is stable. Here u_0 is a root of the equation

$$P\zeta A(1 - u^2)^2 + (k - \varepsilon u)(ku - \varepsilon) = 0$$

or, in the limit as $\varepsilon \to 0$,

$$P\zeta A(1 - u^2) + k^2u = 0 .$$

Further, formulae (40), (51), (53), and (54) yield, when $u = u_0$ as $\varepsilon \to 0$,

$$n(u) = \frac{1}{A}\frac{k - \varepsilon u}{1 - u^2} \to \frac{k}{A}\frac{1}{1 - u^2} ,$$

$$\delta = \frac{P_2'}{P_2} - \frac{P_3'}{P_3} \to -\frac{1}{u} ,$$

$$n' = -\frac{\varepsilon}{A}\frac{P_2}{(1 - u^2)^2} \to \frac{k}{A}\frac{2u}{(1 - u^2)^2} ,$$

$$\delta_1 = \frac{3u}{1 - u^2} + \frac{P_3'}{P_3} \to \frac{3u}{1 - u^2} + \frac{1}{u} ,$$

$$\delta_2 = \frac{1}{4}n'\delta \to -\frac{k}{2A}\frac{1}{(1 - u^2)^2} ,$$

$$\delta_3 = \frac{2}{3}\delta_2\delta_1 \to -\frac{k}{A}\frac{2u^2 + 1}{(1 - u^2)^3} .$$

Note that for $k \neq 0$, δ_2 and δ_3 never vanish. Thus, by theorem 6,

$$[\dot{\psi}] = \frac{k}{A}\frac{1}{1 - u^2} - \frac{k}{2A}\frac{(\Delta u)^2}{(1 - u^2)^2} - \frac{k}{A}\frac{2u^2 + 1}{(1 - u^2)^3}(\Delta u)^3 + O\{(\Delta u)^4\} . \tag{68}$$

This formula can also be obtained directly from the equation of motion of a spherical pendulum (see Sommerfeld, 1947, § 18, formulae (18.8) and (18.9)). To do so, it is necessary to note that Sommerfeld's notation is related to ours in the following manner:

$$\varphi = \psi , \qquad l = \zeta , \qquad C = k/m .$$

Then in our notation his formula (18.8) is

$$\dot{\psi} = \frac{k}{A}\frac{1}{1 - u^2} . \tag{69}$$

Putting this into his formula (18.9), differentiating it, and recalling that

$$-\ddot{\theta}\sin\theta = \ddot{u} + \frac{u\dot{u}^2}{1 - u^2} ,$$

we obtain a second equation

$$\ddot{u} + \frac{u\dot{u}^2}{1 - u^2} + \left(\frac{k}{A}\right)^2 \frac{u}{1 - u^2} + \frac{g}{\zeta}(1 - u^2) = 0 . \tag{70}$$

If we now apply the results of sections 1.1–1.4 of this chapter, we will once again arrive at equation (68) (compare to Tsel'man, 1972).

§ 2. On the Oscillations of a Satellite in the Plane of an Elliptical Orbit

This part of the chapter illustrates the applications of the methods of §§ 3, 4 in chapter III. We consider a nonlinear differential equation describing the planar motion of a satellite with respect to its center of mass, which moves in an elliptical orbit. There are two cases, corresponding to oscillations of the satellite in either the orbital or the absolute system of coordinates. In each case, we will use the calculations of the higher order approximations in the asymptotic method (see example 1, § 4, Chapter III) to refine results on the stability of known periodic solutions, as well as to find new classes of periodic solutions (see § 3, Chapter III).

This problem is interesting because in it we encounter strong degeneracy (to fifth order in a small parameter). The presentation follows that of an earlier work by the author (Bruno, 1976b).

2.1. The Formulas of the Asymptotic Method

We consider the system

$$dX/dt \equiv \dot{X} = \varepsilon F(X, t) \ , \tag{1}$$

where $X = (x_1, \ldots, x_n)$, $F = (f_1, \ldots, f_n)$ are functions analytic with respect to X and t and 2π periodic in t in some domain $\mathscr{H}$. In accordance with the asymptotic method of Krylov and Bogolyubov, as presented by Bogolyubov and Mitropol'skiĭ [1974], there exists a change of coordinates

$$X = Y + \sum_{k=1}^{\infty} \varepsilon^k H_k(Y, t) \ , \tag{2}$$

which transforms system (1) into an "averaged" system

$$\dot{Y} = \sum_{k=1}^{\infty} \varepsilon^k G_k(Y) \equiv G(Y, \varepsilon) \ . \tag{3}$$

Here, the functions $H_k(Y, t)$ are analytic with respect to Y and t and 2π-periodic in t in the domain $\mathscr{H}$. The functions G_k are also analytic there, but do not depend

on t. The series in the right-hand sides of (2) and (3) are formal series (i.e., they can be either convergent or divergent). System (3) is the "normal form" of system (1) (§ 4, Chapter III). From §§ 3, 4 of Chapter III, we know that the normalizing transformation (2) converges on the set of rest points of system (3)

$$\mathscr{A} = \{Y, \varepsilon \colon G(Y, \varepsilon) = 0\} \ , \tag{4}$$

and this set is analytic. The points Y, ε in this set correspond to 2π-periodic solutions of the original system (1). Consequently, to find the 2π-periodic solutions of system (1) for small $|\varepsilon|$, we must calculate the terms in expression (3) which determine the structure of the set (4). To do this, it is not always sufficient to know the first term, G_1. For example, if $G_1 \equiv 0$, then it is necessary to find G_2 and, in those cases when G_2 has multiple roots, even G_3.

In what follows we will be dealing with very degenerate situations and we will have to calculate the terms of expansion (3) to fifth order. We therefore write out the equations which serve to define the functions H_k and G_k ($k = 1, \ldots, 5$) to fifth order as well. These equations are obtained by differentiating equation (2) with respect to t, replacing the derivatives $\dot{X}$ and $\dot{Y}$ according to expressions (1) and (3), and finally expressing X in terms of Y and t according to (2). We obtain

$$F(Y, t) = G_1 + \frac{\partial H_1}{\partial t} \ ,$$

$$\frac{\partial F}{\partial Y} H_1 = G_2 + \frac{\partial H_1}{\partial Y} G_1 + \frac{\partial H_2}{\partial t} \ ,$$

$$\frac{\partial F}{\partial Y} H_2 + \frac{1}{2}\frac{\partial^2 F}{\partial Y^2} H_1^2 = G_3 + \frac{\partial H_1}{\partial Y} G_2 + \frac{\partial H_2}{\partial Y} G_1 + \frac{\partial H_3}{\partial t} \ ,$$

$$\frac{\partial F}{\partial Y} H_3 + \frac{\partial^2 F}{\partial Y^2} H_1 H_2 + \frac{1}{6}\frac{\partial^3 F}{\partial Y^3} H_1^3 = G_4 + \frac{\partial H_1}{\partial Y} G_3 + \frac{\partial H_2}{\partial Y} G_2 + \frac{\partial H_3}{\partial Y} G_1 + \frac{\partial H_4}{\partial t} \ ,$$

$$\begin{aligned} &\frac{\partial F}{\partial Y} H_4 + \frac{1}{2}\frac{\partial^2 F}{\partial Y^2} H_2^2 + \frac{\partial^2 F}{\partial Y^2} H_1 H_3 + \frac{1}{2}\frac{\partial^3 F}{\partial Y^3} H_1^2 H_2 + \frac{1}{24}\frac{\partial^4 F}{\partial Y^4} H_1^4 \\ &= G_5 + \frac{\partial H_1}{\partial Y} G_4 + \frac{\partial H_2}{\partial Y} G_3 + \frac{\partial H_3}{\partial Y} G_2 + \frac{\partial H_4}{\partial Y} G_1 + \frac{\partial H_5}{\partial t} \ . \end{aligned} \tag{5}$$

Here, the partial derivatives with respect to Y and their vector products must be understood in the tensor sense. We replace X with Y as the argument of F.

In what follows, we will treat a very special case of system (1):

$$\dot{x}_1 = \varepsilon x_2 \ , \qquad \dot{x}_2 = \varepsilon f(x_1, t) \ , \tag{6}$$

which is equivalent to a second order equation. For this system, calculations according to formulas (5) yield the following values of $G_k = (g_{1k}, g_{2k})$:

$$
\begin{aligned}
g_{11} &= y_1 , \qquad g_{12} = g_{13} = g_{14} = g_{15} = 0 ; \\
g_{21} &= [f] , \qquad g_{22} = 0 , \\
g_{23} &= \left[\tilde{\tilde{f}}^* \frac{\partial f}{\partial y_1} \right] , \\
g_{24} &= -2y_2 \left[\frac{\partial \tilde{\tilde{\tilde{f}}}^*}{\partial y_1} \frac{\partial f}{\partial y_1} \right] , \\
g_{25} &= 3y_2^2 \left[\frac{\partial^2 \tilde{\tilde{\tilde{f}}}^*}{\partial y_1^2} \frac{\partial f}{\partial y_1} \right] + 3[f] \left[\frac{\partial \tilde{\tilde{\tilde{f}}}^*}{\partial y_1} \frac{\partial f}{\partial y_1} \right] + \left[\left(\widetilde{\widetilde{\tilde{\tilde{f}}^* \frac{\partial f}{\partial y_1}}} \right)^* \frac{\partial f}{\partial y_1} \right] \\
&\quad + \frac{1}{2} \left[\frac{\partial^2 f}{\partial y_1^2} (\tilde{\tilde{f}}^*)^2 \right] .
\end{aligned}
\tag{7}
$$

Here $f = f(y_1, t)$, $[f]$ is its time average, and

$$f^* = f - [f] ,$$

$$\tilde{f}^* = \int f^* \, dt = \sum_{k=1}^{\infty} \frac{1}{k} (a_k \sin kt - b_k \cos kt) ,$$

if

$$f^* = \sum_{k=1}^{\infty} (a_k \cos kt + b_k \sin kt) .$$

Consequently, the "averaged" system for system (6), taken to an accuracy of ε^6, has the form

$$
\begin{aligned}
\dot{y}_1 &= \varepsilon y_2 , \\
\dot{y}_2 &= \varepsilon [f] + \varepsilon^3 \left[\tilde{\tilde{f}}^* \frac{\partial f}{\partial y_1} \right] - \varepsilon^4 2y_2 \left[\frac{\partial \tilde{\tilde{\tilde{f}}}^*}{\partial y_1} \frac{\partial f}{\partial y_1} \right] \\
&\quad + \varepsilon^5 \{ y_2^2 p(y_1) + [f] q(y_1) + r(y_1) \} ,
\end{aligned}
\tag{8}
$$

where

$$r = \left[\left(\widetilde{\widetilde{\tilde{\tilde{f}}^* \frac{\partial f}{\partial y_1}}} \right)^* \frac{\partial f}{\partial y_1} \right] + \frac{1}{2} \left[\frac{\partial^2 f}{\partial y_1^2} (\tilde{\tilde{f}}^*)^2 \right] , \tag{9}$$

We now consider an even more specialized case, when

$$f(y_1, t) = \alpha(t) \sin y_1 + \beta(t) \cos y_1 , \tag{10}$$

where α is an even, and β an odd, function:

$$\alpha(t) = \alpha(-t) , \qquad \beta(t) = -\beta(-t) , \tag{11}$$

and α and β are both 2π-periodic in t. In this case we have

$$[f] = [\alpha]\sin y_1 \ ,$$

$$\left[\tilde{\tilde{f}}^* \frac{\partial f}{\partial y_1}\right] = s\cos y_1 \sin y_1 \ ,$$

$$\left[\frac{\partial \tilde{\tilde{\tilde{f}}}^*}{\partial y_1}\frac{\partial f}{\partial y_1}\right] \equiv 0 \ , \qquad r = r^{(1)}\cos^2 y_1 \sin y_1 + r^{(2)}\sin^3 y_1 \ ,$$

where

$$s = [\alpha\tilde{\tilde{\alpha}}^*] - [\beta\tilde{\tilde{\beta}}] \ ,$$

$$r^{(1)} = [\alpha(\widetilde{\widetilde{\widetilde{\alpha\tilde{\tilde{\alpha}}^*}}})^*] - [\alpha(\widetilde{\widetilde{\widetilde{\beta\tilde{\tilde{\beta}}}}})^*] - [\beta(\widetilde{\widetilde{\widetilde{\alpha\tilde{\tilde{\beta}}}}})] - [\beta\tilde{\tilde{\alpha}}^*\tilde{\tilde{\beta}}] - \tfrac{1}{2}[\alpha(\tilde{\tilde{\beta}})^2] \ , \tag{12}$$

$$r^{(2)} = [\beta(\widetilde{\widetilde{\widetilde{\beta\tilde{\tilde{\alpha}}^*}}})] - \tfrac{1}{2}[\alpha(\tilde{\alpha}^*)^2] \ .$$

Thus, in the case of (10) and (11), the "averaged" system (8) takes the form

$$\begin{aligned} \dot{y}_1 &= \varepsilon y_2 \ , \\ \dot{y}_2 &= \varepsilon[\alpha]\sin y_1 + \varepsilon^3 s\cos y_1 \sin y_1 \\ &\quad + \varepsilon^5\{y_2^2 p(y_1) + [f]q(y_1) + r^{(1)}\cos^2 y_1 \sin y_1 + r^{(2)}\sin^3 y_1\} \ . \end{aligned} \tag{13}$$

2.2. The Problem of the Oscillations of a Satellite

We consider the planar motion of a satellite about its center of inertia, which is in an elliptical orbit in a central gravitational field. Let the principal axis of inertia, with respect to which the moment of inertia is B, always be perpendicular to the plane of the orbit. We denote the moments of inertia with respect to the other principal axes by A and C. The equation of the relative motion was found by Beletskiĭ [1959]. It has the form

$$(1 + e\cos v)\frac{d^2\delta}{dv^2} - 2e\sin v\frac{d\delta}{dv} + \mu\sin\delta = 4e\sin v \ . \tag{14}$$

Here δ is the angle between the radius vector of the center of mass and the principal axis of rotation with moment of inertia C; $\mu = 3(A - C)/B$; e is the eccentricity of the orbit; and v is the angular distance between the radius vector and the perigee of the orbit (the true anomaly). We know that $|A - C| \leqslant B$, so $|\mu| \leqslant 3$.

This nonlinear second-order differential equation (14) has periodic coefficients and two parameters e and μ which lie in the rectangle $\mathscr{E} = \{e, \mu\colon 0 \leqslant e \leqslant 1, |\mu| \leqslant 3\}$. Equation (14) is invariant under the changes

$$\nu \to \nu + \pi \ , \qquad \delta \to \delta \ , \qquad e \to -e \ , \qquad \mu \to \mu \ ; \tag{15}$$

$$\nu \to \nu, \qquad \delta \to \delta + \pi \ , \qquad e \to e \ , \qquad \mu \to -\mu \ ; \tag{16}$$

$$\nu \to -\nu \ , \qquad \delta \to -\delta \ , \qquad e \to e \ , \qquad \mu \to \mu \ . \tag{17}$$

Our task is to study some of the periodic solutions of equation (14) for small $|\mu|$. To do this, we follow Chernousko [1963] and take as new independent variable the time t, reckoned from the perigee, multiplied by 2π and divided by the period of the revolution of the satellite (i.e., t is the mean anomaly):

$$t = 2 \operatorname{Arctg} \sqrt{\frac{1-e}{1+e}} \operatorname{tg} \frac{\nu}{2} - \frac{e\sqrt{1-e^2} \sin \nu}{1 + e \cos \nu} \ , \tag{18}$$

$$t(\nu + 2\pi) = t(\nu) + 2\pi \ .$$

Then

$$dt = (1 - e^2)^{3/2} (1 + e \cos \nu)^{-2} \, d\nu \ .$$

We now introduce new variables x_1 and x_2 according to the formulas of Chernousko [1963]:

$$\delta = mt - 2\nu + x_1 \ , \qquad d\delta/dt = m - 2 + \varepsilon x_2 \ , \tag{19}$$

where $m = \text{const}$ and $\varepsilon = \sqrt{|\mu|}$ are new parameters.

Then equation (14) is equivalent to the system

$$\begin{aligned} \dot{x}_1 &= \varepsilon x_2 \ , \\ \dot{x}_2 &= -\varepsilon \operatorname{sgn} \mu \frac{(1 + e \cos \nu)^3}{(1 - e^2)^3} \sin(mt - 2\nu + x_1) \ , \end{aligned} \tag{20}$$

where ν is considered a function of time and defined by equation (18). If m is an integer, then system (20) has the form (6), (10), (11), where

$$\begin{aligned} \alpha(t) &= -\operatorname{sgn} \mu \frac{(1 + e \cos \nu)^3}{(1 - e^2)^3} \cos(mt - 2\nu) \ , \\ \beta(t) &= -\operatorname{sgn} \mu \frac{(1 + e \cos \nu)^3}{(1 - e^2)^3} \sin(mt - 2\nu) \ . \end{aligned} \tag{21}$$

Chernousko [1963] investigated the first approximation of the averaged system (13) corresponding to various integers m. Here, with the help of higher-order approximations, we consider two cases: $m = 2$ (oscillation in the orbital coordinate system) and $m = 0$ (oscillation in absolute coordinates).

We note that after substitution (15), the right-hand side of the second equation (20) is unchanged for even m but changes sign for odd m. In the averaged system (13), this substitution corresponds to the change

$$t \to t + \pi\ , \qquad e \to -e\ , \qquad Y \to Y\ .$$

Since the right-hand side of system (13) is independent of t, the right-hand side of the second equation (13) must be an odd function of e for odd m and even for even m. Moreover, for any integer $m \neq 2$, the right-hand side of the second equation in the complete averaged system (3) must vanish when $e = 0$; this follows from the integrability of system (20) when $e = 0$.

2.3. Oscillations in the Orbital System of Coordinates

We consider here the case $m = 2$. Here

$$[\alpha] = -\operatorname{sgn}\mu \cdot \varphi(e) \not\equiv 0\ .$$

Chernousko [1963] drew the graph of the function $\varphi(e)$: on the interval $0 \leqslant e \leqslant 1$, $\varphi(e)$ is monotone decreasing and vanishes at $e = e_0 \approx 0.682$.

For $\varepsilon \neq 0$, the rest points of the first approximation of system (13) are determined by the system equations

$$y_2 = 0\ , \qquad \varphi(e)\sin y_1 = 0\ . \tag{22}$$

The solutions of this system with $\mu = 0$ are the generating periodic solutions of system (20); that is, the limits as $\mu \to 0$ of the periodic solutions of system (20). Thus, in accordance with (22), we obtain the following families of generating periodic solutions ($x_2 = 0$ everywhere):

$$x_1 = 0\ , \qquad e - \text{arbitrary}\ ; \tag{23}$$

$$x_1 = \pi\ , \qquad e - \text{arbitrary}\ ; \tag{24}$$

$$x_1 - \text{arbitrary}\ , \qquad e = e_0\ . \tag{25}$$

We will denote families, (23), (24), and (25), of generating periodic solutions by $\mathscr{F}_0$, $\mathscr{F}_1$, and $\mathscr{G}$, respectively. For system (13), the families $\mathscr{F}_0$ and $\mathscr{F}_1$ are

$$\mathscr{F}_0 = \{y_1 = 0, y_2 = 0; e, \mu - \text{arbitrary}\}\ ;$$

$$\mathscr{F}_1 = \{y_1 = \pi, y_2 = 0; e, \mu - \text{arbitrary}\}\ .$$

It follows from property (16) that $\mathscr{F}_0(e, -\mu) = \mathscr{F}_1(e, \mu)$. Consequently, it is sufficient to investigate only one of the families $\mathscr{F}_0$ and $\mathscr{F}_1$.

Now let us disregard terms of higher than third order in ε in system (13) and consider the system

$$\begin{aligned} \dot{y}_1 &= \varepsilon y_2\ , \\ \dot{y}_2 &= -\varepsilon \operatorname{sgn}\mu \cdot \varphi(e)\sin y_1 + \varepsilon^3 s(e)\cos y_1 \sin y_1\ . \end{aligned} \tag{26}$$

Then the fixed points of the family $\mathscr{G}$ satisfy the equation

$$\varphi(e) = \mu s(e) \cos y_1 \ , \tag{27}$$

where the function $s(e)$ is found from formula (12). Since $e = e_0$ is a simple root of the equation $\varphi(e) = 0$, then near that point we can, by the implicit function theorem, solve equation (27) for $e - e_0$:

$$e - e_0 = \zeta(\mu \cos y_1) = \mu \cos y_1 \frac{s(e_0)}{\varphi'(e_0)} + O(\mu^2) \ . \tag{28}$$

That is, for every μ, the family $\mathscr{G}$ exists only for those values of e which lie between boundaries determined from the equation

$$\varphi(e) = \pm s(e)\mu \ . \tag{29}$$

By (28), these boundaries are given approximately by the formula

$$e - e_0 = \pm \left| \mu \frac{s(e_0)}{\varphi'(e_0)} \right| \ . \tag{30}$$

We will now investigate the stability of the rest points of the averaged system (26). The characteristic equation for these solutions is

$$\lambda^2 = -\mu\varphi(e) \cos y_1 + \mu^2 s(e) \cos 2y_1 \ . \tag{31}$$

If $\varphi(e^0) \neq 0$, then near the point $e = e^0$, $\mu = 0$ the stability of solutions of the family $\mathscr{F}_0$ is determined by the sign of the value $\mu\varphi(e^0)$, while the line $\mu = 0$ is the boundary of the region of stability in which $\mu\varphi(e^0) > 0$.

For example, when $\varphi(e^0) > 0$, we will have stability for $\mu > 0$ and instability for $\mu < 0$.

One more boundary of the region of stability of the family $\mathscr{F}_0$ passes through the point $e = e_0$, $\mu = 0$. By (31), this boundary is determined by the equation

$$\varphi(e) = \mu s(e) \ , \tag{32}$$

which is analogous to equation (27) with $y_1 = 0$. Solving this equation, we obtain in accordance with (28),

$$e - e_0 = \zeta(\mu) = \mu \frac{s(e_0)}{\varphi'(e_0)} + O(\mu^2) \ . \tag{33}$$

At the same time, boundary (32) is the intersection of the families $\mathscr{F}_0$ and $\mathscr{G}$. Similarly, the boundary of the region of stability of the family $\mathscr{F}_1$ is its intersection with the family $\mathscr{G}$ and is obtained from equation (27) by letting $\cos y_1 = -1$.

In the family $\mathscr{G}$ we have, according to (27) and (31),

$$\lambda^2 = \mu^2 s(e)(\cos 2y_1 - \cos^2 y_1) = -\mu^2 s(e) \sin^2 y_1 \approx -\mu^2 s(e_0) \sin^2 y_1 \ .$$

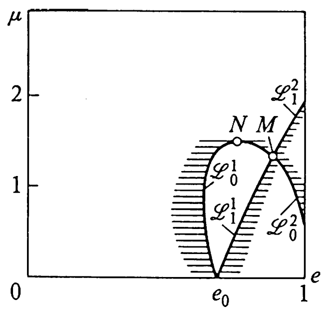

Fig. 86

That is, for small $|\mu|$ the entire family is stable if $s(e_0) > 0$ and unstable if $s(e_0) < 0$. The family $\mathscr{F}_0$ and its stability have been calculated for all values of e, $\mu \in \mathscr{E}$ (see Zlatoustov et al. 1964, Sarychev and Zlatoustov, 1975). The published results are sufficient to show that $s(e_0) > 0$ and that the family $\mathscr{G}$ is stable. In fact, the boundary of the region of stability of the family $\mathscr{F}_0$ is given by equation (33). We see in figure 4 of Zlatoustov et. al. [1964] that $s(e_0)/\varphi'(e_0) < 0$; from figure 3 of Chernousko [1963] we find that $\varphi'(e_0) < 0$; consequently, $s(e_0) > 0$.

Now let us trace the fate of the family $\mathscr{G}$ as μ increases from zero. According to the calculations of Zlatoustov, et al. [1964] and Sarychev and Zlatoustov [1975], the boundaries $\mathscr{L}_0$ and $\mathscr{L}_1$ of the regions of stability of $\mathscr{F}_0$ and $\mathscr{F}_1$, respectively, leave the point $e = e_0$, $\mu = 0$ and then intersect again at a point M (figure 86). The shading shows the location of the regions of stability with respect to its boundaries. The periodic solutions of the families $\mathscr{F}_0$ and $\mathscr{F}_1$ are symmetric in the sense that they transform into themselves under the substitution (17). In contrast, the solutions of family $\mathscr{G}$ are asymmetric.

Equation (14) is conservative and equivalent to a Hamiltonian system if we take the value of one of the parameters e or μ as the Hamiltonian. Hénon [1965] and Bruno [1972b] have considered families of symmetric periodic solutions of a Hamiltonian system with two degrees of freedom. It was shown that, in the general situation, the change of stability (the principal resonance) in a family can be of two types:

I. On a critical solution, the Hamiltonian has an extreme and there is no bifurcation.

II. The Hamiltonian does not have an extreme on a critical solution, and the family generates two families of asymmetric periodic solutions. These families exist only for values of the Hamiltonian which lie to one side of the critical value. If the symmetric family is stable for these values of the Hamiltonian, then the asymmetric families are unstable, and conversely: if the symmetric family is unstable for these values of the Hamiltonian, then the asymmetric families are stable.

The role of the Hamiltonian is played here by the parameter e. For fixed μ, the families $\mathscr{F}_0$ and $\mathscr{F}_1$ pass through points of the curves $\mathscr{L}_0$ and $\mathscr{L}_1$, respectively, and have no extreme values. The point M divides each of the bounding curves $\mathscr{L}_0$ and $\mathscr{L}_1$ into two parts. $\mathscr{L}_0^1$ and $\mathscr{L}_1^1$ are the parts which extend from the point $e = e_0$, $\mu = 0$, while $\mathscr{L}_0^2$ and $\mathscr{L}_1^2$ are the parts of the curves which do not pass through this point. We denote by $\mathscr{G}_j^i$ the families of asymmetric periodic solutions which branch off of the families $\mathscr{F}_0$ and $\mathscr{F}_1$ along the curves $\mathscr{L}_j^i$. Evidently, these families distribute themselves as follows: The families $\mathscr{G}_0^1$ and $\mathscr{G}_1^1$ are defined on that region of the e, μ plane which is bounded by the curves $\mathscr{L}_0^1$ and $\mathscr{L}_1^1$. Together, these families constitute the family $\mathscr{G}$, and are stable throughout that region. Specifically, inside the region the stability index for the family $\mathscr{G}$ is $A < 1$; but there may exist a subregion of a parametric resonance in which $A < -1$ and $\mathscr{G}$ is unstable. After the work of Bruno [1976b], where the family $\mathscr{G}$ was first found, Sarychev, Sazonov and Zlatoustov [1980] computed the entire family $\mathscr{G}$ for $e \leqslant 0.998$. They found that the family exists in the region bounded by the curves $\mathscr{L}_0^1$ and $\mathscr{L}_1^1$, as well outside of that region, and has a fold.

We note that the family $\mathscr{G}$ corresponds to new types of oscillations of the satellite. These are distinguished by the fact that their average values may take on any value between 0 and 2π, while the oscillations of families $\mathscr{F}_0$ and $\mathscr{F}_1$ have the discrete mean values $\delta = 0$ and $\delta = \pi$. In this context, the oscillations of the family $\mathscr{G}$ are stable.

The function $\varphi(e)$ and even the family $\mathscr{F}_0$ have been found for other integers m besides $m = 0$ and $m = 2$ (see Beletskiĭ, 1975, Beletskiĭ and Lavrovsky, 1975, Sarychev, Sazonov, Zlatoustov, 1979). For such values of m, the equation $\varphi(e) = 0$ has one simple root in the interval $0 < e < 1$; $e = e_0$. As above, it appears that in these cases the family (25) generates a family $\mathscr{G}$ of stable periodic solutions. It would be interesting to find the family $\mathscr{G}$ in these cases and to trace its fate outside of the curves $\mathscr{L}_0$ and $\mathscr{L}_1$. It would also be useful to compute the family $\mathscr{F}_0$ for $|\mu| > 3$, in order to discover other curves of bifurcation of other families of periodic solutions. It is possible that such families exist for $|\mu| < 3$ but intersect $\mathscr{F}_0$ only when $|\mu| > 3$.

2.4. Oscillations in the Absolute System of Coordinates

We consider the case $m = 0$. Here $[f] = [\alpha] = 0$, so a first approximation yields nothing. We omit here the calculations of formula (12), which are quite cumbersome (see Bruno, 1977), but they show that

$$s(e) \equiv 0 \ , \tag{34}$$

$$r^{(1)}(e) = r^{(2)}(e) = be^2 \operatorname{sgn}\mu + O(e^4) \ , \qquad b = -\frac{1}{9 \cdot 2^9} \ . \tag{35}$$

Therefore, the averaged system (13), obtained from system (20), has the form

$$\begin{aligned}\dot{y}_1 &= \varepsilon y_2 \ , \\ \dot{y}_2 &= \varepsilon^5\{y_2^2 p(y_1) + r^{(1)}(e)\sin y_1\} \ .\end{aligned} \tag{36}$$

For generating solutions, we obtain the equation

$$r^{(1)}(e)\sin y_1 = 0 \ .$$

Consequently, there are two families

$$\{y_2 = 0, y_1 = 0\} \text{ and } \{y_2 = 0, y_1 = \pi\} \tag{37}$$

as well as the families

$$\{y_1 \text{ is arbitrary, } e \text{ is a root of the equation } r^{(1)}(e) = 0\}$$

According to (35), this equation has the multiple root $e = 0$. Sarychev, Zlatoustov, and Sazonov, [1976, § 3.7 and fig. 30; Sarychev, Sazonov, Zlatoustov, 1977] calculated the function $r^{(1)}(e)$, which they called $A_{31}(e)$, along the interval $0 < e < 1$. According to their calculations, the equation $r^{(1)}(e) = 0$ has no roots in that interval. Hence, for $\mu = 0$, the only generating families are those in (37).

We will denote the families of periodic solutions which are generated by the families (37) as $\mathscr{F}_0$ and $\mathscr{F}_1$, respectively. As a consequence of automorphism (16), $\mathscr{F}_0(e, \mu) = \mathscr{F}_1(e, -\mu)$. Hence, it is sufficient to consider just the family $\mathscr{F}_0$.

For this family of solutions, the characteristic equation of (36) is

$$\lambda^2 = \mu^3 \operatorname{sgn} \mu r^{(1)}(e) \cos y_1 \ .$$

By (21) and (12), the function $r^{(1)} \operatorname{sgn} \mu$ is independent of μ, so that the family $\mathscr{F}_0$ changes its stability when the sign of μ is changed. By (35), $\mathscr{F}_0$ is, for small μ and e, stable when $\mu > 0$ and unstable when $\mu < 0$. Here,

$$|\lambda| \approx \mu^{3/2} e/(3 \cdot 2^4 \sqrt{2}) \ .$$

All of this agrees with the results of the numerical calculations of $\mathscr{F}_0$ and its stability in the work of Zlatoustov, et. al. [1964]. We note that in the case considered, $m = 0$, there is very strong degeneracy (to fifth order in ε).

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Part II
The Sets of Analyticity of a Normalizing Transformation

Preface

Let an analytic system of ordinary differential equations have an invariant irreducible k-dimensional torus $\mathcal{T}$ of conditionally periodic solutions ($k \geqslant 0$). We consider the following problem in a neighborhood of $\mathcal{T}$: find all invariant analytic sets containing $\mathcal{T}$.

A new approach to the problem is proposed in this paper: we construct a formal normalizing transformation (§ 1), and, using the normal form, find the set $\mathcal{M}$ on which the normalizing transformation is analytic (§ 2). The results are as follows: if there are no small divisors ($k \leqslant 1$), then $\mathcal{M} = \mathcal{A}_W$; if small divisors are present, then $\mathcal{M} = \mathcal{B}_W \subset \mathcal{A}_W$. The set $\mathcal{B}_W$ is distinguished from $\mathcal{A}_W$ by the condition of nilpotency of a certain matrix B.

The basic results are then applied to a Hamiltonian system (§ 3). In § 4 and § 5 we find the sets of analyticity for the transformations which normalize the system on an invariant submanifold. It is shown that the set $\mathcal{A}_W$ contains periodic solutions previously found by Poincaré, Lyapunov, Siegel, and others. On $\mathcal{B}_W$ there are the conditionally periodic solutions previously found by Kolmogorov, Arnol'd, Moser, Bogolyubov, and others.

Key words: normal form, periodic solutions, conditionally periodic solutions, Hamiltonian systems.

Table of Contents

Basic Notation

Greek and Latin print letters denote numbers, functions, and series; small letters are scalars; capitals denote vectors and matrices, the components of which are the corresponding small letters, appropriately indexed. A tilde appears underneath those Greek letters which might be confused with Latin letters. Vectors will be written as rows, but behave as column matrices in matrix multiplication.

$\mathbf{i} = +\sqrt{-1}$.

$X = (x_1, \ldots, x_k)$, $Y = (y_1, \ldots, y_{l+m})$,

$V = (v_1, \ldots, v_l)$, $W = (w_1, \ldots, w_m)$

$\Omega = (\omega_1, \ldots, \omega_k)$, the frequency vector.

$\Lambda = (\lambda_1, \ldots, \lambda_l)$ and $\underset{\sim}{M} = (\mu_1, \ldots, \mu_m)$ are vectors of eigenvalues: $\operatorname{Re} \Lambda = 0$, $\operatorname{Re} \mu_j \neq 0$.

$\underset{\sim}{M} = (\underset{\sim}{M}_-, \underset{\sim}{M}_+)$, where $\operatorname{Re} \underset{\sim}{M}_- < 0 < \operatorname{Re} \underset{\sim}{M}_+$.

$P = (p_1, \ldots, p_k)$, $Q = (q_1, \ldots, q_l)$.

E_j is the jth unit vector,

E is the identity matrix.

$\underset{\sim}{E} = (\varepsilon_1, \ldots, \varepsilon_n)$ is a vector of parameters.

$\|Q\| = q_1 + \cdots + q_l$, and

$|Q| = |q_1| + \cdots + |q_l|$.

$Q' = (q_1, \ldots, q_{l'})$, $Q'' = (q_{l'+1}, \ldots, q_l)$.

$Q \geqslant 0$ means that $q_j \geqslant 0$ for all $j = 1, \ldots, l$.

$\bar{x}$ is the complex conjugate to a number x.

$\dot{x} = dx/dt$.

$\langle Q, \Lambda \rangle = q_1 \lambda_1 + \cdots + q_l \lambda_l$, the scalar product.

$V^Q = v_1^{q_1} v_2^{q_2} \ldots v_l^{q_l}$.

$\mathbb{C}^m$ is m-dimensional complex space.

$\mathbb{Z}^m$ is the m-dimensional integer lattice.

$\mathbb{N}_j^l = \{Q \colon Q \in \mathbb{Z}^l, Q + E_j \geqslant 0\}$, and

$\mathbb{N}^l = \mathbb{N}_1^l \cup \cdots \cup \mathbb{N}_l^l$.

The remaining sets are denoted by Latin script capitals:

$\mathscr{T}$ is the k-dimensional initial torus.

$\mathscr{P}_X$ is the ring of analytic, 2π-periodic functions of a vector variable X.

$\mathscr{P}_X[[Y]]$ is the ring of formal power series in Y with variable coefficients belonging to $\mathscr{P}_X$.

$\mathscr{P}_X^{o}[[Y]]$ is the ring of convergent series in $\mathscr{P}_X[[Y]]$.
$\mathscr{V}$, $\mathscr{W}_-$, and $\mathscr{W}_+$ are integral manifolds of a seminormal form:
$\mathscr{V} = \mathscr{W}_- \cap \mathscr{W}_+ = \{U, V, W, \underset{\sim}{E}: W = 0\}$.
$\mathscr{A}_W$ and $\mathscr{B}_W$ are formal sets, $\mathscr{A}$ and $\mathscr{B}$ their projections onto $\mathscr{V}$.
$\mathscr{K} = \{U, V, \underset{\sim}{E}: V'' = 0\}$ is a coordinate subspace of the V coordinates.
$\mathscr{L} = \mathscr{L}[\lambda_1, \ldots, \lambda_{l'}]$ is an invariant coordinate subspace.
$\hat{\Psi} = \Psi|_{V''=0}$,
$\mathscr{A}' = \mathscr{A} \cap \mathscr{K}$, $\mathscr{B}'' = \mathscr{B} \cap \mathscr{K}$.
$\mathscr{F}_\sigma^j$ is a σ-parameter family of j-dimensional tori.

Introduction

We shall consider the system

$$dZ/dt \equiv \dot{Z} = F(Z, \underset{\sim}{E}) \ , \tag{0.1}$$

where $Z = (z_1, \ldots, z_{k+l+m})$ is a vector variable and $\underset{\sim}{E} = (\varepsilon_1, \ldots, \varepsilon_n)$ is an vector of parameters. Let this system (0.1) be analytic in some domain $\mathscr{G} \subset \mathbb{C}^{k+l+m+n}$ and real in the domain $\operatorname{Re}\mathscr{G}$. Suppose that in the domain $\operatorname{Re}\mathscr{G}$, with $\underset{\sim}{E} = 0$, system (0.1) has an invariant, irreducible k-dimensional torus $\mathscr{T}$ filled by conditionally periodic solutions. The following problem arises:

Problem 0.1. *For system* (0.1) *in the neighborhood of the torus* $\mathscr{T}$, *find all invariant sets* $\mathscr{M}$ *which contain* $\mathscr{T}$.

This problem can be approached on two levels:

1) The formal-algebraic level, in which we seek formal power series which describe the invariant set $\mathscr{M}$.

2) The analytic level, in which we investigate the convergence of those power series.

In turn, the first level has two aspects:

a) the complex case, in which we seek invariant sets $\mathscr{M}$.

b) the real case, in which we investigate the real parts $\operatorname{Re}\mathscr{M}$ of complex invariant sets $\mathscr{M}$.

Note that the classical problem of the continuation of the torus $\mathscr{T}$ for a small parameter ε is really a special case of problem 0.1. The question of bifurcations of families of such solutions under a variation of parameters is likewise a special case of this problem. Considerable effort has been devoted to solving special cases of problem 0.1.

We present here a new approach to this problem: we construct a formal normalizing transformation for system (0.1) and then find the sets of analyticity of that transformation. In §§ 1 and 2, we present this approach in greater detail and formulate some fundamental theorems. In §§ 2–5, we derive various special results which follow from these general theorems. Nearly all of the references

treat these special results. On the one hand, there are theorems on the continuation and existence of periodic solutions and on bifurcations of families of such solutions (Poincaré, Lyapunov, Siegel, Meyer, Henrard, etc.). On the other hand, there are theorems on the existence of conditionally periodic solutions (Kolmogovov, Arnold, Moser, Bogolyubov, etc.). This second group of results is connected with the problem of the effects of "small divisors" which are not addressed at all by the first group.

Let us briefly describe our new approach. We limit ourselves to considering a torus $\mathscr{T}$ for which the system on the torus and the variational system are reducible (restrictions 1 and 2), and such that among the eigenvalues of the variational system there are l with vanishing real part (denoted by $\lambda_1, \ldots, \lambda_l$) and m with non-zero real part (denoted by $\mu_1, \ldots, \mu_m$). We construct a formal transformation which puts the original system into a seminormal form, which becomes a normal form when $m = 0$. We use a variant of the seminormal form which 1) facilitates our investigation of the real parts of the invariant sets and 2) gives us the analyticity of the formal sets under minimal requirements. We pose a second problem for the seminormalizing transformation:

Problem 2.2. *Find the sets of analyticity of such a transformation.*

The answer to this problem is something like this: In the absence of small divisors, the transformation converges on some set $\mathscr{A}_W$ which is composed of stationary, periodic, and asymptotic solutions. If there are small divisors, then the transformation converges on some subset $\mathscr{B}_W$ of $\mathscr{A}_W$. The set $\mathscr{B}_W$ consists of conditionally periodic and asymptotic solutions. In this connection, the sets $\mathscr{A}_W$ and $\mathscr{B}_W$ are defined by equations which are derived from the normal form. The set $\mathscr{A}_W$ is described by $k + l$ equations, while the subset $\mathscr{B}_W$ is distinguished by the fact that a certain matrix B of order $k + l$ is nilpotent on $\mathscr{B}_W$. Therefore the fixed and periodic solutions are easily continued with respect to a parameter, while the continuation of conditionally periodic solutions requires either a large number of parameters or that the system possess some property which guarantees the nilpotency of the matrix B. Invertible and Hamiltonian systems have such properties; so most results on the existence of conditionally periodic solutions apply to Hamiltonian systems. The main results of this work are briefly given in the article [Bruno, 1975].

Before we proceed with our systematic presentation, let us make two remarks:

1. It is often said that the conditionally periodic solutions on a k-dimensional torus $\mathscr{T}$ constitute a k-parameter family of solutions. Since each of those solutions is everywhere dense on $\mathscr{T}$, the parametrization of solutions is not regular. Hence, such a set of solutions cannot be called a family. The term family is reserved for a collection of invariant sets in which the dependence on parameters has a regular character (here, analytic or formal). We consider here only families of invariant tori, none of which is called a family if it is irreducible.

2. Some of the results discussed here were initially obtained for analytic systems, but were than carried over to sufficiently smooth systems (0.1) [Moser, 1962; Kelley, 1967a, 1967b, 1968a; Rüssmann, 1970; Bogolyubov, Mitropolsky, Samoilenko, 1969]. Apparently, the following general principle is valid: for every invariant analytic set of an analytic system, there is a corresponding smooth invariant set of a smooth system. It thus appears that all the results presented here are equally applicable to sufficiently smooth systems.

§ 1. The Seminormal Form

1.1. Statement of the Problem

Suppose the autonomous system

$$\dot{Z} = F(Z, \Delta) \ , \tag{1.1}$$

where $Z = (z_1, \ldots, z_{k+l+m})$ is a vector variable and $\Delta = (\delta_1, \ldots, \delta_n)$ is a vector of parameters ($\dot{\Delta} = 0$), defined and analytic on some domain $\mathscr{G}$ of the $(k + l + m + n)$ dimensional space $\mathbb{C}^{k+l+m+n}$ and real in $\operatorname{Re}\mathscr{G}$. Let the k-dimensional torus $\mathscr{T} \subset \operatorname{Re}\mathscr{G}$ be an irreducible integral manifold of system (1.1) for $\Delta = \Delta^o$. That is, the torus $\mathscr{T}$ consists of complete solutions of system (1.1), any of which is everywhere dense on $\mathscr{T}$. In what follows, we will call such an integral manifold *an initial torus* [Bruno, 1973]. Our task (problem 0.1) is to find, in a sufficiently small neighborhood $\mathscr{U} \subset \mathscr{G}$ of the torus $\mathscr{T}$, all integral analytic sets of system (1.1) which contain $\mathscr{T}$. Our attention will mainly be focused on those real, integral, analytic sets which pass through $\mathscr{T}$. Our presentation has been designed to facilitate the study of these real sets.

1.2. Local Coordinates

We now introduce local parameters, $\underset{\sim}{E} = (\varepsilon_1, \ldots, \varepsilon_n) = \Delta - \Delta^o$. We will call functions $x_1(Z, \underset{\sim}{E}), \ldots, x_k(Z, \underset{\sim}{E}), y_1(Z, \underset{\sim}{E}), \ldots, y_{l+m}(Z, \underset{\sim}{E})$, defined and analytic in some complex neighborhood of $\mathscr{T}$, *local coordinates* if, on the torus $\mathscr{T}$ itself:

1) $y_1 = \cdots = y_{l+m} = 0$;
2) the Jacobian, $\det(\partial(X, Y)/\partial Z)$, does not vanish;
3) the functions x_j have period 2π: $x_j + 2\pi = x_j, j = 1, \ldots, k$.

Then the neighborhood of the torus $\mathscr{T}$ in the region $\operatorname{Re}\mathscr{G}$ is a fibre bundle of the torus $\{X \bmod 2\pi\}$ on the $(l + m)$-dimensional ball $\{|Y| < \varepsilon_0\}$ and the n-dimensional ball $\{|\underset{\sim}{E}| < \varepsilon_0\}$; in this neighborhood, the torus is defined by the equations $Y = 0$, $\underset{\sim}{E} = 0$. Such local coordinates exist for every torus $\mathscr{T} \subset \operatorname{Re}\mathscr{G}$.

In these local coordinates, system (1.1) becomes

$$\begin{aligned} \dot{X} &= \Theta^{(1)}(X, Y, \underset{\sim}{E}) \ , \\ \dot{Y} &= \Theta^{(2)}(X, Y, \underset{\sim}{E}) \ , \end{aligned} \tag{1.2}$$

where the $\Theta^{(j)}$ are functions which are analytic for $|Y| < \varepsilon_0$, $|\underset{\sim}{E}| < \varepsilon_0$ and 2π-periodic in X. Since the torus $\mathscr{T}$ is an invariant manifold, $\Theta^{(2)}(X,0,0) \equiv 0$ and on the torus $\mathscr{T}$ system (1.2) has the form

$$\dot{X} = \Theta^{(1)}(X,0,0) \ . \tag{1.3}$$

By supposition, all solutions of system (1.3) are conditionally periodic with k basic frequencies. For example, the system

$$\dot{X} = \Omega = (\omega_1, \ldots, \omega_k) = \text{const} \tag{1.4}$$

has such solutions if

$$\langle P, \Omega \rangle \neq 0 \text{ for all } P \subset \mathbb{Z}^k \ , \qquad P \neq 0 \ . \tag{1.5}$$

Restriction 1. *Let system* (1.3) *take the form of system* (1.4) *in suitable local coordinates.*

As a consequence of the irreducibility of solutions on the torus $\mathscr{T}$, property (1.5) will be satisfied. If $k > 1$, then such coordinates X do not always exist on $\mathscr{T}$. From now on, however, we will only consider initial tori which satisfy restriction 1. Note that when $k = 0$ ($\mathscr{T}$ is a stationary point) or $k = 1$ ($\mathscr{T}$ is a periodic solution), the restriction is automatically satisfied.

We now consider an approximation, linear in Y, to the second subsystem of (1.2) for $\underset{\sim}{E} = 0$:

$$\dot{Y} = A(X)Y \ , \qquad (A = \partial\Theta^{(2)}/\partial Y \text{ at } Y = 0, \underset{\sim}{E} = 0) \ . \tag{1.6}$$

This is the so-called variational system for the integral manifold $\mathscr{T}$.

Restriction 2. *In suitable local coordinates, let the matrix $A(X)$ in system* (1.6) *be constant.*

This restriction might be replaced by the weaker requirement of analytic reducibility of system (1.6) to some "linear normal form". But we will continue to use restriction 2 for the sake of simplicity. In this connection, we can assume that the constant matrix A has already been put into some normal form – say, Jordan form. The coordinates Y can be complex for real Z.

If $k = 0$, then $\mathscr{T}$ is a point $Z = Z^0$, local coordinates are $Y = Z - Z^o$, and the matrix A is always constant. If $k = 1$, $\mathscr{T}$ is a periodic solution, and restriction 2 is always satisfied for complex coordinates, Y, but not always for real Y [this is the Floquet-Lyapunov theorem; see Pontryagin, 1961, § 19]. For $k > 1$, restriction 2 is generally not automatically satisfied, even in complex coordinates.

Thus, for an initial torus $\mathscr{T}$ which satisfies restrictions 1 and 2, system (1.1) becomes, in suitable coordinates,

$$\begin{aligned} \dot{X} &= \Omega + \underset{\sim}{Z}^{(1)}(X, Y, \underset{\sim}{E}) \ , \\ \dot{Y} &= AY + \underset{\sim}{Z}^{(2)}(X, Y, \underset{\sim}{E}) \ , \end{aligned} \tag{1.7}$$

where the functions $\underset{\sim}{Z}^{(1)}$ and $\underset{\sim}{Z}^{(2)}$ are analytic for $|Y|, |\underset{\sim}{E}|, |\mathrm{Im}\, X| < \varepsilon_0$, the matrix A has normal form, and

$$\underset{\sim}{Z}^{(1)} = 0 \ , \qquad \underset{\sim}{Z}^{(2)} = 0 \ , \qquad \partial \underset{\sim}{Z}^{(2)}/\partial Y = 0 \text{ at } Y = 0 \ , \qquad \underset{\sim}{E} = 0 \ .$$

Moreover, real values of the coordinates Z and Δ correspond to real values of X and $\underset{\sim}{E}$ and to those values of the Y coordinates which satisfy the reality relation

$$\bar{Y} = KY \ ,$$

where K is some constant matrix and $\bar{Y}$ denotes the complex conjugate coordinates to the Y coordinates. The reality of the original system (1.1) implies the following properties:

$$\begin{aligned} \bar{\underset{\sim}{Z}}^{(1)}(X, KY, \underset{\sim}{E}) &= \underset{\sim}{Z}^{(1)}(X, Y, \underset{\sim}{E}) \ , \\ K^{-1}\bar{A}K &= A \ , \\ K^{-1}\bar{\underset{\sim}{Z}}^{(2)}(X, KY, \underset{\sim}{E}) &= \underset{\sim}{Z}^{(2)}(X, Y, \underset{\sim}{E}) \ . \end{aligned} \tag{1.7'}$$

1.3. Analytic and Formal Entities

Let a function $f(X)$ be analytic and 2π-periodic for sufficiently small $|\mathrm{Im}\, X|$. It can be expanded in a Fourier series

$$f = \sum_{P \in \mathbb{Z}^k} f_P \exp \mathrm{i}\langle P, X\rangle \ ,$$

where the coefficients f_P decrease like $\exp(-c|P|)$ as $|P|$ increases, where c is some positive constant. All such functions f form a ring, which we denote by $\mathscr{P}_X$. This ring is closed with respect to differentiation; that is, if $f \in \mathscr{P}_X$, then $\partial f/\partial x_j \in \mathscr{P}_X$. We denote by $\mathscr{P}_X[[Y]]$ the ring of formal power series

$$f = \sum_{0 \leqslant Q \in \mathbb{Z}^{l+m}} f_Q(X) Y^Q \ , \tag{1.8}$$

where $Y^Q = y_1^{q_1} \dots y_{l+m}^{q_{l+m}}$ and $f_Q \in \mathscr{P}_X$. Clearly, every series in this ring has a unique Taylor-Fourier expansion

$$f = \sum f_{PQ} Y^Q \exp \mathrm{i}\langle P, X\rangle \ .$$

We will call series (1.8) *convergent* if there exists some positive number $\varepsilon_0 > 0$ such that the series

$$\sum |f_Q| |y_1|^{q_1} \dots |y_{l+m}|^{q_{l+m}}$$

converges for all X and Y in the neighborhood

$$|\mathrm{Im}\, X| < \varepsilon_0 \ , \qquad |Y| < \varepsilon_0 \ . \tag{1.9}$$

The convergent power series (1.8) also constitute a ring, which we denote by $\mathscr{P}_X^0[[Y]]$. Both rings, $\mathscr{P}_X^0[[Y]]$ and $\mathscr{P}_X[[Y]]$, are closed relative to differentia-

tion with respect to both X and Y. The sum $f(X, Y)$ of a convergent series (1.8) is analytic and 2π-periodic in X in the neighborhood (1.9).

A set $\mathscr{M}$ of points X, Y is called an *analytic set* on the torus $Y = 0$ if there exists a neighborhood (1.9) in which the set $\mathscr{M}$ coincides with the set of solutions of some system of equations

$$f_1 = \cdots = f_r = 0 \ , \tag{1.10}$$

where $f_j \in \mathscr{P}_X^0[[Y]]$ and $f_j = 0$ at $Y = 0$ [see Fuchs, 1961]. An ideal $\mathscr{J}$ with basis $f_1, \ldots, f_r$ in the ring $\mathscr{P}_X^0[[Y]]$ corresponds to $\mathscr{M}$. If $f_j \in \mathscr{P}_X[[Y]]$, we say that the system of equations (1.10) defines a *formal set* $\mathscr{M}$ for which there is a corresponding ideal $\mathscr{J}$, with basis $f_1, \ldots, f_r$, in $\mathscr{P}_X[[Y]]$. The set $\mathscr{M}$ will be analytic if there is a basis $\tilde{f}_1, \ldots, \tilde{f}_r \in \mathscr{P}_X^0[[Y]]$. A power series $h \in \mathscr{P}_X[[Y]]$ is said to be *convergent on a formal set* $\mathscr{M}$ if there exists a convergent series $g \in \mathscr{P}_X^0[[Y]]$ such that $h - g \in \mathscr{J}$.

The set $\mathscr{M}$ is called a (local) *manifold* if, under a suitable indexing of the Y variables, the set is defined by a system of equations

$$y_j = g_j(X, y_1, \ldots, y_{m'}) \ , \qquad j = m' + 1, \ldots, l + m \ ,$$

where the $g_j \in \mathscr{P}_X[[y_1, \ldots, y_{m'}]]$ and $m' \leqslant l + m$. A manifold $\mathscr{M}$ is either analytic or formal, depending on whether the series g_j are convergent or not.

Returning to system (1.7), we note that its right-hand sides belong to $\mathscr{P}_X^0[[Y, \underset{\sim}{E}]]$. A set is determined by the system (1.10), where the $f_j \in \mathscr{P}_X[[Y, \underset{\sim}{E}]]$. A formal set (1.10) is said to be an *integral* (or invariant) if

$$\left\langle \frac{\partial f_j}{\partial X}, \underset{\sim}{Z}^{(1)} \right\rangle + \left\langle \frac{\partial f_j}{\partial Y}, \underset{\sim}{Z}^{(2)} \right\rangle = 0 \ , \qquad j = 1, \ldots, r \ , \tag{1.10'}$$

on that set; that is, if in the ideal $\mathscr{J}$, the equations (1.10′) are identically satisfied. In particular, a formal manifold is determined by a system of the form

$$\begin{aligned} y_i &= g_i \ , \qquad i = m' + 1, \ldots, l + m \ , \\ \varepsilon_j &= h_j \ , \qquad j = n' + 1, \ldots, n \ , \end{aligned} \tag{1.10''}$$

where $g_i, h_j \in \mathscr{P}_X[[y_1, \ldots, y_{m'}; \varepsilon_1, \ldots, \varepsilon_{n'}]]$. This will be an integral manifold for system (1.7) if, after making the substitutions (1.10″), the equations

$$\zeta_i^{(2)} = \left\langle \frac{\partial g_i}{\partial X}, \underset{\sim}{Z}^{(1)} \right\rangle + \sum_{k=1}^{m'} \frac{\partial g_i}{\partial y_k} \zeta_k^{(2)} \ , \qquad i = m' + 1, \ldots, l + n$$

$$0 = \left\langle \frac{\partial h_j}{\partial X}, \underset{\sim}{Z}^{(1)} \right\rangle + \sum_{k=1}^{m'} \frac{\partial h_j}{\partial y_k} \zeta_k^{(2)} \ , \qquad j = n' + 1, \ldots, n$$

become identities in the ring $\mathscr{P}_X[[y_1, \ldots, y_{m'}; \varepsilon_1, \ldots, \varepsilon_{n'}]]$.

1.4. The Normal Form

Among the $l + m$ eigenvalues of the matrix A let there be l of them, $\lambda_1, \ldots, \lambda_l$, with zero real part ($\operatorname{Re} \lambda_i = 0$) and m of them, $\mu_1, \ldots, \mu_m$, with non-zero real part ($\operatorname{Re} \mu_j \neq 0$). In this section we consider the case $m = 0$, so that $\lambda_1, \ldots, \lambda_l$ are all of the eigenvalues of A. Since A has a normal form, its main diagonal is $\Lambda = (\lambda_1, \ldots, \lambda_l)$. We now try, with the help of a nonlinear change of local coordinates, to simplify system (1.7). We make a formal change of coordinates

$$\begin{aligned} X &= U + \Xi(U, V, \underset{\sim}{E}) , \\ Y &= V + \underset{\sim}{H}(U, V, \underset{\sim}{E}) , \end{aligned} \tag{1.11}$$

where the series $\xi_i, \eta_j \in \mathscr{P}_U[[V, \underset{\sim}{E}]]$ and $\Xi = 0$, $\underset{\sim}{H} = 0$, and $\dfrac{\partial \underset{\sim}{H}}{\partial V} = 0$ at $V = 0$, $\underset{\sim}{E} = 0$. Such a formal transformation is invertible. It takes system (1.7) into a formal system

$$\begin{aligned} \dot{U} &= \Phi(U, V, \underset{\sim}{E}) , \\ \dot{V} &= \Psi(U, V, \underset{\sim}{E}) , \end{aligned} \tag{1.12}$$

where $\varphi_i, \psi_j \in \mathscr{P}_U[[V, \underset{\sim}{E}]]$; $\Phi(U, 0, 0) = \Omega$; $\Psi(U, 0, 0) = 0$; and $\partial\Psi/\partial V = A$ at $V = 0$, $\underset{\sim}{E} = 0$. We set

$$g_j = \psi_j / v_j = \sum g_{jPQS} V^Q \underset{\sim}{E}^S \exp \mathbf{i}\langle P, U\rangle \ ; \tag{1.13}$$

here, P runs over $\mathbb{Z}^k$, S runs over the first orthant of $\mathbb{Z}^n$ ($S \geqslant 0$), and Q runs over a set $\mathbb{N}_j^l \subset \mathbb{Z}^l$ such that

$$Q + E_j \geqslant 0 \ , \qquad \|Q\| \geqslant 0 \ , \qquad j = 1, \ldots, l$$

[see Bruno, 1971, introduction]. We write $\mathbb{N}^l = \mathbb{N}_1^l \cup \cdots \cup \mathbb{N}_l^l$. For Φ we use the usual expansion

$$\Phi = \sum \Phi_{PQS} V^Q \underset{\sim}{E}^S \exp \mathbf{i}\langle P, U\rangle \ , \tag{1.14}$$

where $P \in \mathbb{Z}^k$, $0 \leqslant Q \in \mathbb{Z}^l$, $0 \leqslant S \in \mathbb{Z}^n$. Recall that the components of the vector Λ are the eigenvalues of the matrix A and that Λ is the principal diagonal of A. We will call system (1.12) a *normal form*, if, in expansions (1.13) and (1.14), the only non-zero coefficients g_{jPQS} and Φ_{PQS} are those for which

$$\mathbf{i}\langle P, \Omega\rangle + \langle Q, \Lambda\rangle = 0 \ . \tag{1.15}$$

Let $Q \in \mathbb{N}^l$ be a fixed vector. We define

$$\gamma(Q) = \underline{\lim} \frac{\ln|\mathbf{i}\langle P, \Omega\rangle + \langle Q, \Lambda\rangle|}{|P|} \ ,$$

where the lim inf is taken for all $P \in \mathbb{Z}^k$ which do not satisfy equation (1.15) as $|P| \to \infty$.

Restriction 3. $\gamma(Q) \geqslant 0$ *for all* $Q \in \mathbb{N}^l$.

Note that for $k = 0$ or $k = 1$, this restriction is met automatically. But when $k > 1$ and $l \geqslant 1$, this is an added restriction on the vectors Ω and Λ. In some cases this restriction can be replaced by a weaker restriction 3′ (see § 5).

Theorem 1.1. *For system* (1.7) *with* $m = 0$, *there exists a formal transformation* (1.11) *to the normal form* (1.12).

For the proof when $k = 0$, see Bruno [1964; 1971, § 1]; for $k = 1$, see Bruno [1971, § 11]. For $k > 1$, the proof is similar. We now note a few properties of the normal form and the normalizing transformation.

1. The advantage of the normal form (1.12) over the original system (1.7) is that expansions (1.13) and (1.14) contain only resonant terms. Hence, the solution of system (1.12) reduces to the integration of a system of reduced order [see Bruno, 1971, § 2].

More specifically, there exists a change of coordinates of the form

$$\tilde{v}_i = v_1^{\kappa_{i1}} \dots v_l^{\kappa_{il}} \exp \mathrm{i}\langle \underset{\sim}{N}_i, U\rangle \ , \qquad i = 1, \dots, l \tag{1.15′}$$

with rational κ_{ij} and ν_{ij}, which transforms the normal form (1.12) into a system

$$\begin{aligned} \dot{U} &= \tilde{\Phi}(\tilde{v}_{l-d+1}, \dots, \tilde{v}_l, \underset{\sim}{E}) \ , \\ (\ln \tilde{v}_j)^{\bullet} &= \tilde{g}_j(\tilde{v}_{l-d+1}, \dots, \tilde{v}_l, \underset{\sim}{E}) \ , \qquad j = 1, \dots, l \ , \end{aligned} \tag{1.15″}$$

where d is the number of linearly independent solutions $Q \in \mathbb{Z}^l$, $P \in \mathbb{Z}^k$ of equation (1.15). The solution of system (1.15″) reduces to the solution of the subsystem of order d defined by the last d equations. For every solution of this subsystem, the values of the variables U and $\tilde{v}_1, \dots, \tilde{v}_{l-d}$ are found by quadratures. If we denote by $\tilde{\lambda}_j$ the constant term in $\tilde{g}_j$, then

$$\tilde{\Lambda} = \underset{\sim}{K}\Lambda + \mathrm{i}\underset{\sim}{N}\Omega \ , \tag{1.15‴}$$

where we have introduced the matrices $\underset{\sim}{K} = (\kappa_{ij})$ and $\underset{\sim}{N} = \begin{pmatrix} \underset{\sim}{N}_1 \\ \vdots \\ \underset{\sim}{N}_l \end{pmatrix} = (\nu_{ij})$.

Also $\tilde{\lambda}_{l-d+1} = \dots = \tilde{\lambda}_l = 0$. Generally speaking, transformation (1.15) is defined away from the coordinate subspaces (that is, for all $v_j \neq 0$).

2. If, under a suitable indexing of the variables, the inequalities

$$\lambda_j \neq \mathrm{i}\langle P, \Omega\rangle + q_1\lambda_1 + \dots + q_{l'}\lambda_{l'} \ , \qquad j = l' + 1, \dots, l \tag{1.15⁗}$$

are satisfied for some $l' < l$ and for all $P \in \mathbb{Z}^k$, and for all integers $q_1, \dots, q_{l'} \geqslant 0$,

then the coordinate subspace

$$v_{l'+1} = \cdots = v_l = 0$$

is an integral manifold of the normal form (1.12). We denote this manifold by $\mathscr{L} = \mathscr{L}[\lambda_1, \ldots, \lambda_{l'}]$; it is a formal integral manifold of system (1.7).

3. Generally speaking, the normalizing transformation (1.11) is not unique [Bruno, 1971, § 1].

4. If system (1.7) satisfies the reality conditions (1.7′) then there exists a normalizing transformation which preserves the reality relations

$$\bar{U} = U \ , \qquad \bar{V} = KV \ , \qquad \bar{\underset{\sim}{E}} = \underset{\sim}{E}$$

in system (1.12).

5. Automorphisms. If system (1.7) is unchanged by the linear change of variables

$$\tilde{X} = M^{(1)}X \ , \qquad \tilde{Y} = M^{(2)}Y \ , \qquad \tilde{t} = \delta t \ ,$$

then there exists a normal form (1.12) which is unchanged by the analogous change of variables

$$\tilde{U} = M^{(1)}U \ , \qquad \tilde{V} = M^{(2)}V \ , \qquad \tilde{t} = \delta t \ .$$

In particular, the normal form of an invertible system (1.7) is likewise invertible.

6. Small parameters $\underset{\sim}{E}$ do not change under a normalizing transformation [see Bruno, 1973, 1974b]. They can be considered local coordinates satisfying the trivial system $\dot{\underset{\sim}{E}} = 0$.

7. If condition (1.5) is not fulfilled, theorem 1.1 still holds. That is, it is possible to put a system into normal form in the neighborhood of any invariant torus which satisfies restrictions 1, 2, and 3. Also, the initial torus may be stratified into invariant tori of lower dimension. It is necessary only that motion on the torus be described by a system (1.4), with Ω arbitrary. However, in this work we always assume that (1.5) is satisfied.

8. Likewise, if $m > 0$, then it is possible to normalize system (1.7) if restriction 3 is met. Here, Λ is the vector of all the eigenvalues of the matrix A. Such a transformation yields the greatest possible simplification of the original system (1.7), but it converges under extremely strict conditions. Therefore, when $m > 0$, we will put the system into a seminormal form which is not as simple as the normal form but for which the seminormalizing transformation converges under less strict conditions [this was observed by Bruno, 1971, § 9].

Example 1.1. Let us find conditions on Ω, λ_i and μ_j under which the normal form is necessarily a linear system

$$\dot{U} = \Omega \ , \qquad \dot{\check{V}} = A\check{V} \ ,$$

where by $\check{V}$ we mean the vector $(v_1, \ldots, v_{l+m})$. Clearly, property (1.5) must be satisfied, and equation (1.15), with

$$\Lambda = \check{\Lambda} = (\lambda_1, \ldots, \lambda_l, \mu_1, \ldots, \mu_m)$$

can have no solutions $P \in \mathbb{Z}^k$, $Q \in \mathbb{N}^{l+m}$, $\|Q\| \geqslant 1$. For this to be true, it is sufficient that any $Q \in \mathbb{N}^{l+m}$, $\|Q\| \geqslant 1$ satisfy the inequality

$$\langle Q, \operatorname{Re} \check{\Lambda} \rangle \neq 0 \ .$$

If $l = 0$ and $\operatorname{Re} \mu_1, \ldots, \operatorname{Re} \mu_m < 0$, this can be assured by requiring that

$$\operatorname{Re}(\mu_i + \mu_j - \mu_k) < 0 \ , \qquad i, j, k = 1, \ldots, m \ .$$

1.5. The Seminormal Form

We now consider the case $m > 0$. Let m_- and m_+ be the numbers of eigenvalues, $\mu_1, \ldots, \mu_m$, with negative and positive real parts, respectively $(m_- + m_+ = m)$. Since the matrix A has normal form, then it has block-diagonal form after a suitable indexing of variables Y:

$$A = \{A_0, A_-, A_+\} \ .$$

where the blocks A_0, A_-, and A_+ are matrices, the eigenvalues of which have zero, negative, and positive real parts, respectively. Moreover, these eigenvalues lie on the diagonals of the respective blocks:

$$\operatorname{diag} A_0 = \Lambda = (\lambda_1, \ldots, \lambda_l) \ ,$$

$$\operatorname{diag} A_- = \underset{\sim}{M}_- = (\mu_1, \ldots, \mu_{m_-}) \ ,$$

$$\operatorname{diag} A_+ = \underset{\sim}{M}_+ = (\mu_{m_-+1}, \ldots, \mu_m) \ .$$

We will divide every vector $W = (w_1, \ldots, w_m)$ into two parts: $W_- = (w_1, \ldots, w_{m_-})$ and $W_+ = (w_{m_-+1}, \ldots, w_m)$.

We now apply to system (1.7) a formal change of variables

$$\begin{aligned} X &= U + \Xi(U, V, W, \underset{\sim}{E}) \ , \\ y_i &= v_i + \eta_i(U, V, W, \underset{\sim}{E}) \ , \qquad i = 1, \ldots, l \ , \\ y_{l+j} &= w_j + \eta_{l+j}(U, V, W, \underset{\sim}{E}) \ , \qquad j = 1, \ldots, m \ , \end{aligned} \tag{1.16}$$

where $V = (v_1, \ldots, v_l)$, $W = (w_1, \ldots, w_m)$, and all the series ξ_i and η_j belong to $\mathscr{P}_U[[V, W, \underset{\sim}{E}]]$; also, $\Xi = 0$, $\underset{\sim}{H} = 0$, $\partial \underset{\sim}{H}/\partial V = 0$, $\partial \underset{\sim}{H}/\partial W = 0$ at $V = 0$, $W = 0$, $\underset{\sim}{E} = 0$. Such a formal transformation is invertible. It takes system (1.7) into a formal system

$$
\begin{aligned}
\dot{U} &= \Phi(U, V, \underset{\sim}{E}) + \check{\Phi}(U, V, W, \underset{\sim}{E}) \ , \\
\dot{V} &= \Psi(U, V, \underset{\sim}{E}) + \check{\Psi}(U, V, W, \underset{\sim}{E}) \ , \\
\dot{W}_- &= \underset{\sim}{X}_-(U, V, W_-, \underset{\sim}{E}) + \check{\underset{\sim}{X}}_-(U, V, W, \underset{\sim}{E}) \ , \\
\dot{W}_+ &= \underset{\sim}{X}_+(U, V, W_+, \underset{\sim}{E}) + \check{\underset{\sim}{X}}_+(U, V, W, \underset{\sim}{E}) \ ,
\end{aligned}
\tag{1.17}
$$

where all the series in the right-hand sides belong to $\mathscr{P}_U[[V, W, \underset{\sim}{E}]]$. Recalling that $W = (W_-, W_+)$, we take

$$
\begin{aligned}
\check{\Phi} = 0 \ , \qquad \check{\Psi} &= 0 \quad \text{for } W = 0 \ ; \\
\check{\underset{\sim}{X}}_- &= 0 \quad \text{for } W_+ = 0 \ ; \\
\check{\underset{\sim}{X}}_+ &= 0 \quad \text{for } W_- = 0 \ ;
\end{aligned}
\tag{1.18}
$$

and $\Phi = \Omega$, $\Psi = 0$, $\underset{\sim}{X} = 0$, $\dfrac{\partial \Psi}{\partial V} = A_0$, $\dfrac{\partial \underset{\sim}{X}}{\partial V} = 0$, and $\dfrac{\partial \underset{\sim}{X}}{\partial W} = \{A_-, A_+\}$ when $V = 0$, $W = 0$, and $\underset{\sim}{E} = 0$.

The series Φ and Ψ in (1.17) are independent of W; they can thus still be expanded as (1.13) and (1.14). The series $\underset{\sim}{X}_-$ and $\underset{\sim}{X}_+$ are independent of W_+ and W_-, respectively; let us consider the expansion

$$
\chi_j = w_j h_j = w_j \sum h_{jPQRS} V^Q W^R \underset{\sim}{E}^S \exp \mathbf{i}\langle P, U\rangle \ , \qquad j = 1, \dots, m \ , \tag{1.19}
$$

where $P \in \mathbb{Z}^k$, $0 \leqslant Q \in \mathbb{Z}^l$. $0 \leqslant R \in \mathbb{N}_j^m$, and $0 \leqslant S \in \mathbb{Z}^n$.

We call the system (1.17), (1.18) a *seminormal form* if:

1) The expansions of Φ and Ψ have the properties of a normal form; that is, the coefficients in the series (1.13) and (1.14) vanish except when their indices satisfy equation (1.15).

2) $\check{\Phi} = 0$, $\check{\Psi} = 0$, $\underset{\sim}{X}_- = \check{\underset{\sim}{X}}_- = 0$ for $W_- = 0$;
$\check{\Phi} = 0$, $\check{\Psi} = 0$, $\underset{\sim}{X}_+ = \check{\underset{\sim}{X}}_+ = 0$ for $W_+ = 0$.

3) The coefficient h_{jPQRS} in expansion (1.19) vanishes unless $\langle R, \operatorname{Re} \underset{\sim}{M}\rangle = 0$; that is,

$$
\begin{aligned}
\langle R_-, \operatorname{Re} \underset{\sim}{M}_-\rangle = 0 \ , \qquad R_+ &= 0 \quad \text{for } j \leqslant m_- \ , \\
R_- = 0 \ , \qquad \langle R_+, \operatorname{Re} \underset{\sim}{M}_+\rangle &= 0 \quad \text{for } j > m_- \ .
\end{aligned}
\tag{1.20}
$$

The seminormal form for $m_+ = 0$ and $k = 0$ was introduced by the author [Bruno, 1971, § 9]; the normal form is a special case of that situation.

Theorem 1.2. *For a system* (1.7) *which meets restriction* 3, *there exist a formal transformation* (1.16) *to a seminormal form* (1.17), (1.18).

We note a few properties of the seminormal form (1.17):

1. The following formal manifolds are integral manifolds

$$\mathscr{W}_- = \{U, V, W, \underset{\sim}{E} \colon W_+ = 0\} \ ,$$

$$\mathscr{W}_+ = \{U, V, W, \underset{\sim}{E} \colon W_- = 0\} \ ,$$

$$\mathscr{V} = \{U, V, W, \underset{\sim}{E} \colon W = 0\} \ .$$

2. Clearly, $\mathscr{V} = \mathscr{W}_- \cap \mathscr{W}_+$. On the integral manifold $\mathscr{V}$, system (1.17) reduces to system (1.12), a normal form.

3. On the manifold $\mathscr{W}_-$, system (1.17) becomes

$$\begin{aligned} \dot{U} &= \Phi(U, V, \underset{\sim}{E}) \ , \\ \dot{V} &= \Psi(U, V, \underset{\sim}{E}) \ , \\ \dot{W}_- &= \underset{\sim}{X}_-(U, V, W_-, \underset{\sim}{E}) \ , \end{aligned} \tag{1.21}$$

which is a seminormal form in the sense of §9 [Bruno, 1971]. Let the variables W be ordered in such a manner that

$$\operatorname{Re}\mu_1 \leqslant \cdots \leqslant \operatorname{Re}\mu_{m_-} < 0 < \operatorname{Re}\mu_{m_-+1} \leqslant \cdots \leqslant \operatorname{Re}\mu_m \ . \tag{1.22}$$

Then in system (1.21), the χ_i depend only on those w_j such that either $i < j \leqslant m_-$ or $j \leqslant i$ and $\operatorname{Re}\mu_j = \operatorname{Re}\mu_i$. That is, $\underset{\sim}{X}_-$ has quasitriangular form [see Bruno, 1971, §2]. Specifically, if for some $j < m_-$

$$\operatorname{Re}\mu_j < \operatorname{Re}\mu_{j+1} \ , \tag{1.23}$$

then the manifold

$$\mathscr{W}_j = \{U, V, W, \underset{\sim}{E} \colon W_+ = 0, w_{j+1} = \cdots = w_{m_-} = 0\}$$

is a subset of $\mathscr{W}_-$ and is an integral manifold for system (1.21) and, consequently, for system (1.17).

Similarly, for any $j > m_-$ for which equation (1.23) holds, there is a formal integral manifold of system (1.17),

$$\mathscr{W}_j = \{U, V, W, \underset{\sim}{E} \colon W_- = 0, w_{m_-+1} = \cdots = w_j = 0\} \ ,$$

which lies on $\mathscr{W}_+$. Clearly, the following relationships hold:

$$\mathscr{W}_i \subset \mathscr{W}_j \subset \mathscr{W}_- \ , \qquad 0 < i < j \leqslant m_- \ ;$$

$$\mathscr{W}_+ \supset \mathscr{W}_i \supset \mathscr{W}_j \ , \qquad m_- < i < j \leqslant m \ .$$

4. The seminormal form still has all properties of reality and invariance with respect to linear automorphisms which the original system (1.7) might have possessed (see properties 4 and 5 in section 1.4).

5. The real parts of the manifolds $\mathscr{W}_-$ and $\mathscr{W}_+$ are filled with solutions that approach the manifold $\mathscr{V}$ as $t \to +\infty$ and $t \to -\infty$, respectively.

6. Let $\mathcal{M}$ be some set on the manifold $\mathcal{V}$. We take

$$\mathcal{M}_W = \{U, V, W, \underset{\sim}{E}: U, V, \underset{\sim}{E} \in \mathcal{M}\} ,$$

$$\mathcal{M}_- = \mathcal{M}_W \cap \mathcal{W}_- , \qquad \mathcal{M}_+ = \mathcal{M}_W \cap \mathcal{W}_+ ,$$

$$\mathcal{M}_j = \mathcal{M}_W \cap \mathcal{W}_j .$$

If the set $\mathcal{M}$ is invariant for system (1.12), then $\mathcal{M}_-$, $\mathcal{M}_+$, and $\mathcal{M}_j$ will be invariant sets of system (1.17). Generally speaking, however, $\mathcal{M}_W$ will not be an invariant set. The solutions in the sets Re $\mathcal{M}_-$ and Re $\mathcal{M}_+$ asymptotically approach solutions in $\mathcal{M}$ as $t \to +\infty$ and $t \to -\infty$, respectively.

In the following analysis the role of the W coordinates is insignificant, and we recommend that they be ignored (assume $m = 0$) on first reading. Fundamental problems are solved in the U, V, $\underset{\sim}{E}$ coordinates, without reference to the W coordinates.

We will call system (1.12), obtained from the seminormal form (1.17) by setting $W = 0$, the normal form of the original system (1.7) with $m \neq 0$.

§ 2. Questions of Convergence

2.1. Convergence in an Entire Neighborhood

If a normalizing transformation (1.11) (or seminormalizing transformation (1.16)) converges for sufficiently small $|\mathrm{Im}\, U|$, $|V|$, $|W|$, and $|\underset{\sim}{E}|$, then it establishes in some neighborhood of the initial torus, a one-to-one correspondence between the solutions of the original system (1.7) and those of the simplified system (1.12) (or (1.17)). However, there does not always exist such a neighborhood of the initial torus $\mathscr{T}$, i.e., a neighborhood in which the transformation mentioned converges. There thus arises:

Problem 2.1. *Under what conditions on the seminormal form* (1.17) *does the normalizing transformation converge in some neighborhood of the initial torus*?

A more traditional statement of the problem asks about the convergence of the normalizing transformation in the sense of property 8 of section 1.4.

Conditions on the normal form which guarantee the convergence of the normalizing transformation have been found for $k = 0$ and $k = 1$, and it has been shown that these cannot be improved upon [Bruno, 1967, 1969, 1971]. For $k > 1$, only the problem of transformation to a linear system has been considered [Belaga, 1962; Bogolyubov, Mitropolskiĭ, Samoilenko, 1969, Ch. VI, theorem 25], as in example 1.1.

The problem of the convergence of the seminormalizing transformation for $k = 0$ and $m_+ = 0$ has been addressed by the author [Bruno, 1971, § 9]. Transformations to forms which possess individual properties of the seminormal form have been treated by Kelley [1967b, 1968b, 1969] and Bibikov [1970a, 1970b, 1973b].

We now formulate an answer to problem 2.1. We take

$$\alpha_j = \min|\mathbf{i}\langle P, \Omega\rangle + \langle Q, \Lambda\rangle| \text{ for } P \in \mathbb{Z}^k ,$$

$$Q \in \mathbb{N}^l , \qquad \|Q\| \geqslant 0 , \qquad \mathbf{i}\langle P, \Omega\rangle + \langle Q, \Lambda\rangle \neq 0 ,$$

$$|P| + \|Q\| < 2^j , \qquad j = 1, 2, \ldots$$

We then write

$$\alpha = \inf \alpha_j \ , \qquad j = 1, 2, \dots ;$$

$$\beta = \sum_{j=1}^{\infty} \frac{\ln \alpha_j}{2^j} \ .$$

Condition α: $\alpha > 0$.

Condition β: $\beta > -\infty$.

If condition α is satisfied, then "small divisors" are absent. Condition α is equivalent to saying that the numbers $\mathbf{i}\omega_1, \dots, \mathbf{i}\omega_k, \lambda_1, \dots, \lambda_l$ are mutually comparable. If condition β is satisfied, then there may be small divisors, but they are not too small. If condition α is fulfilled, then so is condition β, since $\beta \geqslant \ln \alpha > -\infty$. In § 5 of Bruno [1971] condition α is called condition ε. Condition β for $k = 0$ was introduced by the author [Bruno, 1967] as "condition α", and later [Bruno, 1971] for $k = 0$ and $k = 1$ as "condition ω".

Normally, we place limits on the smallness of divisors by employing Siegel's condition, which can be written in our notation as

$$\alpha_j > c2^{-\nu j} \ , \qquad j = 1, 2, \dots , \tag{2.1}$$

where c and ν are positive constants. If this condition is fulfilled, then

$$\beta > \sum_{j=1}^{\infty} \frac{\ln c - \nu j \ln 2}{2^j} = \ln c - \nu \ln 2 \sum_{j=1}^{\infty} \frac{j}{2^j} > -\infty \ ;$$

that is, condition β is fulfilled. We also note that under condition β, restriction 3 is automatically met.

Condition A. *There exists a series* $a(U, V, \underset{\sim}{E}) \in \mathscr{P}_U[[V, \underset{\sim}{E}]]$ *such that in system* (1.17)

$$\varphi_i = \omega_i a(U, V, \underset{\sim}{E}) \ , \qquad i = 1, \dots, k \ ,$$

$$\psi_j \equiv v_j g_j = v_j \lambda_j a(U, V, \underset{\sim}{E}) \qquad j = 1, \dots, l \ .$$

Theorem 2.1. *If, in an analytic system* (1.7), *Ω and Λ satisfy condition β and if the seminormal form* (1.17) *satisfies condition A, then the seminormalizing transformation* (1.16) *is analytic in some complex neighborhood of the initial torus $\mathscr{T}$.*

This theorem is proved for $k = 0$ and $m_+ = 0$ in § 9 of Bruno [1971] and is formulated for $k = 1$ and $m = 0$ in § 11 of the same work.

2.2. Convergence on Sets

Note that condition β is an extremely weak limitation, since it is satisfied by almost all Ω and Λ [Siegel, 1952]. In contrast, condition A imposes an extremely

strict limitation on the right-hand sides of the seminormal form, and requires $k+l-1$ linear relationships among the series Φ and Ψ. The author has shown [Bruno, 1971, theorem III] that condition A cannot be weakened. We therefore pose a second problem.

Problem 2.2: *For a seminormal form* (1.17), *find all sets* $\mathscr{M}$ *which are analytic on the initial torus* $\mathscr{T}$ *and on which the seminormalizing transformation* (1.16) *converges for any given analytic system* (1.7).

Let us clarify this. As we know, the set of points of convergence of a power series of one variable, $f(y)$, consists of a disk $|y| < \rho$ and some of the points of its circumference $|y| = \rho$. The set of points of convergence is of a more complicated nature for a power series in several variables. For example, the series

$$y_1 \sum_{k=0}^{\infty} k!\, y_2^k$$

converges only on the axes $y_1 = 0$ and $y_2 = 0$ and diverges at any point $y_1 \neq 0$, $y_2 \neq 0$. In the terminology of section 1.3, problem 2.2 can be formulated as follows: for a seminormal form (1.17), we seek those formal sets $\mathscr{M}$ on which the series ξ_i and η_i in the seminormalizing transformation converge. The convergence of the series Ξ and H from (1.16) and Φ, Ψ, and $\underset{\sim}{X}$ from (1.17) on a set $\mathscr{M}$ (or modulo its ideal, $\mathscr{J}$) is proved all at once; as a consequence, we obtain the analyticity of the set $\mathscr{M}$ itself.

We will consider the equalities of condition A as equations which define a formal set $\mathscr{A}_W$. That is,

$$\mathscr{A}_W = \{U, V, W, \underset{\sim}{E} \colon \Phi = \Omega a, \psi_j = \lambda_j v_j a, j = 1, \ldots, l\} . \tag{2.2}$$

Here, a behaves as a free parameter. It can be eliminated from the equations, so that $\mathscr{A}_W$ is defined by the system

$$\begin{aligned} &\frac{\varphi_i}{\omega_i} = \frac{\psi_j}{\lambda_j v_j} , && i = 1, \ldots, k , \\ &\lambda_j \neq 0 , && 1 \leqslant j \leqslant l ; \\ &\psi_{j'} = 0 , && \text{if } \lambda_{j'} = 0 . \end{aligned} \tag{2.3}$$

Theorem 2.2. *If condition* α *is satisfied for system* (1.7), *then the seminormalizing transformation* (1.16) *is analytic on the set* $\mathscr{A}_W$, *which is itself analytic.*

In this theorem, condition α excludes the presence of small divisors; under condition (1.5) this is possible only if $k = 0$ or $k = 1$. We now consider the case of small divisors. Let $L = \{\lambda_1, \ldots, \lambda_l\}$ be the diagonal matrix formed from the diagonal of the matrix A_0. We consider the square matrix of order $k + l$

$$B = \begin{pmatrix} \dfrac{\partial \Phi}{\partial U} & \dfrac{\partial \Phi}{\partial V} \\ & \\ \dfrac{\partial \Psi}{\partial U} & \dfrac{\partial \Psi}{\partial V} - La \end{pmatrix} \tag{2.4}$$

on the set $\mathscr{A}_W$; here a is just the parameter which appears in the equations (2.2) which define the set $\mathscr{A}_W$. We define the formal set $\mathscr{B}_W$, a subset of $\mathscr{A}_W$, as that set on which the matrix B is nilpotent. That is

$$\mathscr{B}_W = \{U, V, W, \underset{\sim}{E}: U, V, W, \underset{\sim}{E} \in \mathscr{A}_W, B^{k+l} = 0\} .$$

Theorem 2.3. *If condition β is satisfied for system* (1.7), *then the semi-normalizing transformation* (1.16) *is analytic on the set $\mathscr{B}_W$, which is itself analytic.*

Thus, in the presence of small divisors, the nilpotency of the matrix B is necessary for our transformations to converge. The author has noted the necessity of this sort of condition in the problem of a normalizing transformation [Bruno, 1967, condition A_1; see also Bruno, 1971, §8].

Note that both sets $\mathscr{A}_W$ and $\mathscr{B}_W$ are defined only with respect to U, V, and $\underset{\sim}{E}$; the W coordinates can be arbitrary on those sets. It is therefore sufficient to investigate the properties of the sets

$$\mathscr{A} = \mathscr{A}_W \cap \mathscr{V} \text{ and } \mathscr{B} = \mathscr{B}_W \cap \mathscr{V} .$$

Example 2.1. We show here that theorem 2.1 is a special case of theorem 2.3; that is, under condition A, the matrix B is nilpotent for every U, V, and $\underset{\sim}{E}$. In fact, under condition A,

$$\frac{\partial \varphi_i}{\partial U} = \omega_i \frac{\partial a}{\partial U} , \qquad \frac{\partial \varphi_i}{\partial V} = \omega_i \frac{\partial a}{\partial V} ,$$

$$\frac{\partial \psi_j}{\partial U} = \lambda_j v_j \frac{\partial a}{\partial U} , \qquad \frac{\partial \psi_j}{\partial V} = \lambda_j a E_j + \lambda_j v_j \frac{\partial a}{\partial V} ,$$

that is,

$$B = \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_k \\ \lambda_1 v_1 \\ \vdots \\ \lambda_l v_l \end{pmatrix} \left(\frac{\partial a}{\partial u_1} \cdots \frac{\partial a}{\partial u_k} \frac{\partial a}{\partial v_1} \cdots \frac{\partial a}{\partial v_l} \right) .$$

Consequently,

$$\begin{aligned} B^2 &= B \left(\sum_{i=1}^{k} \frac{\partial a}{\partial u_i} \omega_i + \sum_{j=1}^{l} \frac{\partial a}{\partial v_j} \lambda_j v_j \right) \\ &= B \sum_{P,Q,S} (\mathrm{i}\langle P, \Omega \rangle + \langle Q, \Lambda \rangle) a_{PQS} V^Q \underset{\sim}{E}^S \exp \mathrm{i}\langle P, U \rangle , \end{aligned} \tag{2.5}$$

where $a = \sum a_{PQS} V^Q E^S \exp \mathbf{i}\langle P, U \rangle$ is the power series expansion of a. Since this expansion contains only resonant terms for which the exponents P and Q satisfy equation (1.15), it follows that the sum (2.5) over P, Q, and S vanishes identically. Consequently, $B^2 = 0$ for any U, V, and $\underset{\sim}{E}$.

Example 2.2. In connection with the preceding example, we might ask: is it possible that $\mathscr{B} \neq \mathscr{A}$? We show here that it is possible. We consider the normal form

$$\dot{u}_1 = 1 \ ,$$

$$\dot{u}_2 = \sqrt{2} \ ,$$

$$\dot{v}_1 = v_1(v_1 - \varepsilon_1) \ .$$

For this system, $k = 2$, $l = 1$, $m = 0$, $n = 1$, $\lambda_1 = 0$, and the set $\mathscr{A}$ is defined by the equation $v_1(v_1 - \varepsilon_1) = 0$; that is, it consists of two components

$$\mathscr{A}^1 = \{U, v_1, \varepsilon_1 \colon v_1 = 0\} \qquad \mathscr{A}^2 = \{U, v_1, \varepsilon_1 \colon v_1 = \varepsilon_1\} \ .$$

Here, the matrix B is

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2v_1 - \varepsilon_1 \end{pmatrix}$$

The set $\mathscr{B}$ is distinguished from the set $\mathscr{A}$ by the additional equation $2v_1 - \varepsilon_1 = 0$. Examining this equation on each of the components $\mathscr{A}^1$ and $\mathscr{A}^2$, we see that $\mathscr{B} = \{U, v_1, \varepsilon_1 \colon v_1 = \varepsilon_1 = 0\}$. That is, $\mathscr{B} = \mathscr{T} \neq \mathscr{A}$.

2.3. Properties of the Set $\mathscr{A}$

We consider here the properties of the set $\mathscr{A} = \mathscr{A}_W \cap \mathscr{V}$. It is defined by the normal form (1.12), which, for simplicity, we will assume is analytic.

2.3.1. General Properties

1) Let

$$b(U, V, \underset{\sim}{E}) = \sum b_{PQS} V^Q \underset{\sim}{E}^S \exp \mathbf{i}\langle P, U \rangle$$

be a series containing only resonant terms. That is, $b_{PQS} \neq 0$ only when equation (1.15) holds for the indices P and Q. We will show that, on the set $\mathscr{A}$,

$$\dot{b} = 0 \ .$$

In fact,

$$\dot{b} = \sum_{i=1}^{k} \frac{\partial b}{\partial u_i} \varphi_i + \sum_{j=1}^{l} \frac{\partial b}{\partial v_j} \psi_j \ .$$

On the set $\mathscr{A}$, we have

$$\begin{aligned}\dot{b} &= a\sum_{i=1}^{k}\frac{\partial b}{\partial u_i}\omega_i + a\sum_{j=1}^{l}\frac{\partial b}{\partial v_j}\lambda_j v_j\\ &= a\sum_{P,Q,S}(\mathrm{i}\langle P,\Omega\rangle + \langle Q,\Lambda\rangle)b_{PQS}V^Q\underset{\sim}{E}^S\exp\mathrm{i}\langle P,U\rangle\ .\end{aligned}$$

Since P and Q always satisfy equation (1.15), the term in parentheses vanishes. The assertion is proved.

2) The set $\mathscr{A}$ is invariant in system (1.12). In fact, by (2.3), the set $\mathscr{A}$ is defined by equations of the form

$$\varphi_i/\omega_i = g_j/\lambda_j \quad \text{and} \quad g_{j'} = 0\ .$$

We need to verify that differentiation with respect to t does not cause these equations to be violated on $\mathscr{A}$. But all of these series contain only resonant terms. Thanks to the preceding property, we know that the derivatives of such series vanish on the set $\mathscr{A}$. Consequently, on $\mathscr{A}$,

$$\dot{\varphi}_i/\omega_i = \dot{g}_j/\lambda_j = 0 \quad \text{and} \quad \dot{g}_{j'} = 0\ .$$

3) The value of the parameter a is fixed on each solution in $\mathscr{A}$. In fact, $a = \varphi_i/\omega_i$ on $\mathscr{A}$, and, as we showed above, $\dot{a} = 0$ there.

4) Let a point U^0, V^0, $\underset{\sim}{E}^0$ be in $\mathscr{A}$, and let $a = a^0$ at this point. From the preceding property, we know that any solution which passes through the point U^0, V^0, $\underset{\sim}{E}^0$ satisfies the system

$$\dot{U} = \Omega a^0\ , \qquad \dot{v}_j = \lambda_j v_j a^0\ , \qquad j = 1,\dots,l\ .$$

From this we obtain

$$\begin{aligned} U &= \Omega a^0 t + U^0\ ,\\ v_j &= v_j^0\exp(\lambda_j a^0 t)\ , \qquad j = 1,\dots,l\ . \end{aligned} \tag{2.6}$$

5) Let d be the number of linearly independent solutions $P \in \mathbb{Z}^k$, $Q \in \mathbb{Z}^l$ of equation (1.15). In the set

$$\mathscr{V}^* = \{U, V, \underset{\sim}{E}\colon v_1 \neq 0,\dots,v_l \neq 0\}$$

we can, in accordance with property 1 of section 1.4 introduce coordinates $\tilde{V}$, such that the normal form (1.12) takes the form (1.15″). Then the set $\mathscr{A}^* = \mathscr{A} \cap \mathscr{V}^*$ is defined by the system of equations

$$\begin{aligned} \tilde{\Phi}(\tilde{v}_{l-d+1},\dots,\tilde{v}_l,\underset{\sim}{E}) &= \Omega a\ ,\\ \tilde{g}_j(\tilde{v}_{l-d+1},\dots,\tilde{v}_l,\underset{\sim}{E}) &= \tilde{\lambda}_j a\ , \qquad j = 1,\dots,l\ . \end{aligned} \tag{2.7}$$

Since $\tilde{\lambda}_{l-d+1} = \cdots = \tilde{\lambda}_l = 0$, then we can separate from system (2.7) a subsystem

$$\tilde{g}_j(\tilde{v}_{l-d+1},\ldots,\tilde{v}_l,\underset{\sim}{E}) = 0 \ , \qquad j = l-d+1,\ldots,l \ . \tag{2.8}$$

That is, rest points of the subsystem of the last d equations of system (1.15″) correspond to solutions in the set $\mathscr{A}^*$. This is not true for all rest points of this subsystem, but only for those at which the functions $\tilde{\Phi}$ and $\tilde{g}_1,\ldots,\tilde{g}_{l-d}$ take on the values Ωa and $\tilde{\lambda}_1 a,\ldots,\tilde{\lambda}_{l-d}a$, respectively. Thus, the set $\mathscr{A}^*$ is defined by $k+l$ equations in the $d+n+1$ variables $\tilde{v}_{l-d+1},\ldots,\tilde{v}_l$, $\underset{\sim}{E}$, and a; that is, there are

$$\sigma = d + n + 1 - (k + l)$$

free parameters on the set $\mathscr{A}^*$. If $\sigma \leqslant 0$, then $\mathscr{A}^*$ is the empty set; if $\sigma > 0$, then $\mathscr{A}^*$ is a σ-parameter family of solutions. Note that σ can always be made positive by increasing n, the number of parameters.

6) In the preceding arguments, we tacitly assumed that among the numbers $\omega_1,\ldots,\omega_k,\lambda_1,\ldots,\lambda_l$, at least one was non-zero, and that the parameter a was defined on the set $\mathscr{V}^*$. We now consider the case $k=0$, $\lambda_1 = \cdots = \lambda_l = 0$; that is, $d = l$. In this case we can assume that $a \equiv 1$, and the set $\mathscr{A}$ is defined by $k + l = d$ equations in $d + n$ unknowns. That is, the number of free parameters on $\mathscr{A}$ is

$$\sigma = n = \dim \mathscr{A} \ .$$

The set $\mathscr{A}$ itself consists of the rest points.

2.3.2. The Set $\mathscr{A}$ on Subspaces

Let $0 \leqslant l' \leqslant l$; we divide every vector $V = (v_1,\ldots,v_l)$ of dimension l into two subvectors $V' = (v_1,\ldots,v_{l'})$ and $V'' = (v_{l'+1},\ldots,v_l)$ of dimensions of l' and $l-l'$ respectively. Let

$$\mathscr{K} = \{U, V, \underset{\sim}{E}\colon V'' = 0\} \tag{2.8'}$$

be a coordinate subspace of a manifold $\mathscr{V}$. Along with the set $\mathscr{A}$ we will also consider the set $\mathscr{A}' = \mathscr{A} \cap \mathscr{K}$. If equations (1.15⁗) are satisfied, then $\mathscr{K}$ is an integral manifold of system (1.12), which in this case we will denote by $\mathscr{L}$.

7) The intersection of $\mathscr{A}$ with any coordinate subspace $\mathscr{K}$ is invariant in system (1.12). This is evident from solution (2.6). The set $\mathscr{A}' = \mathscr{A} \cap \mathscr{K}$ is given by the system of equations

$$\hat{\Phi} = \Omega a \ , \tag{2.9}$$

$$\hat{\psi}_j = \lambda_j v_j a \ , \qquad j = 1,\ldots,l' \ , \tag{2.10}$$

$$\hat{\Psi}'' = 0 \ , \tag{2.11}$$

where "$\hat{\ }$" means that the functions are evaluated at $V'' = 0$. Let d be the number of linearly independent solutions $P \in \mathbb{Z}^k$, $Q' \in \mathbb{Z}^{l'}$ of the equation

$$\mathrm{i}\langle P, \Omega\rangle + \langle Q', \Lambda'\rangle = 0 \ .$$

We write

$$\mathscr{K}^* = \{U, V, \underset{\sim}{E}\colon V'' = 0; v_j \neq 0, j = 1, \ldots, l'\} \ .$$

According to (2.6), solutions from the set $\mathscr{A}'^* = \mathscr{A}' \cap \mathscr{K}^*$ have $r' = k + l - d$ independent frequencies. We call the number $r' = r'(\mathscr{K})$ the *irrationality index of the subspace* $\mathscr{K}$. By virtue of (1.5),

$$k \leqslant r'(\mathscr{K}) \leqslant k + l' \ .$$

If $\mathscr{K}^{(1)} \supset \mathscr{K}^{(2)}$, then $r'(\mathscr{K}^{(1)}) \geqslant r'(\mathscr{K}^{(2)})$.

8) On the integral manifold $\mathscr{L}$, we have $\hat{\Psi}'' \equiv 0$. Therefore, the set $\mathscr{A}' = \mathscr{A} \cap \mathscr{L}$ is defined by the system (2.9), (2.10) above. Let us apply a transformation to the V' variables like that in the discussion of property 1, § 1.4. We find that the number of parameters on the set $\mathscr{A}'^* = \mathscr{A}' \cap \mathscr{L}^*$ is

$$\sigma' = \begin{cases} d' + n + 1 - (k + l') = n + 1 - r' \ , & \text{if } r' \neq 0 \ , \\ n \ , \quad \text{if } r' = 0 \ . \end{cases}$$

Specifically, we always have that $\sigma' \leqslant n$; that is, for "general systems" the set $\mathscr{A}$ reduces to the initial torus $\mathscr{T}$ unless the number n of parameters is positive. If $\mathscr{L}^1 \supset \mathscr{L}^2$, then

$$\sigma'(\mathscr{L}^1) \leqslant \sigma'(\mathscr{L}^2) \ ,$$

hence, it always happens that $\sigma \leqslant \sigma'$.

2.3.3. The Real Part

We now consider the real part of the set $\mathscr{A}$, (see property 4 in § 1.4).

The parameter a takes real values on the set $\operatorname{Re}\mathscr{A}$.

9) All solutions in the set $\operatorname{Re}\mathscr{A}$ are conditionally periodic, as may be seen from (2.6). If this solution lies in the subspace $\mathscr{K}$ but not in any smaller subspace, then it lies on an r'-dimensional invariant torus ($r' = r'(\mathscr{K})$), and has r' independent frequencies. In particular, solutions in $\operatorname{Re}\mathscr{A}^*$ have $r(\mathscr{V})$ independent frequencies.

10) Thus, the set $\operatorname{Re}\mathscr{A}$ consists of invariant tori filled by conditionally periodic solutions. As may be seen from property 5, the torus $\mathscr{T}^0 \subset \mathscr{A}$ is defined by the constant values $\tilde{v}^0_{l-d+1}, \ldots, \tilde{v}^0_l, \underset{\sim}{E}^0$, which correspond to the value $a = a^0$. The frequencies on this torus $\mathscr{T}^0$ are

$$\Omega a^0, \mathrm{i}\tilde{\lambda}_1 a^0, \ldots, \mathrm{i}\tilde{\lambda}_{l-d} a^0 \ .$$

The eigenvalues of the torus $\mathscr{T}^0$ are the eigenvalues of the rest point, $\tilde{v}^0_{l-d+1}, \ldots, \tilde{v}^0_l, \underset{\sim}{E}^0$ in the subsystem of the last d equations of the system (1.15″), i.e., the eigenvalues of the matrix

$$\frac{\partial(\tilde{g}_{l-d+1}, \ldots, \tilde{g}_l)}{\partial(\ln \tilde{v}_{l-d+1}, \ldots, \ln \tilde{v}_l)} \tag{2.11'}$$

evaluated at the indicated rest point.

Note that the real dimension of the set Re $\mathscr{A}$ does not exceed the complex dimension of the set $\mathscr{A}$, but may be less. In particular, the number of real parameters on the set Re $\mathscr{A}'$ does not exceed the number σ' defined in property 8.

2.3.4. Adjoining Families of Tori

A $(j + s)$-dimensional set is called an s-parameter *family* of j-dimensional irreducible invariant tori of the system (1.12) and is denoted by $\mathscr{F}_s^j$ if on this set there are independent functions $h_1, \ldots, h_s$ (parameters) to each value of which, $h_1^0, \ldots, h_s^0$ there corresponds a j-dimensional invariant torus in $\mathscr{F}_s^j$. The family $\mathscr{F}_s^j$ *adjoins* the torus $\mathscr{T}$ if the intersection of the family with any neighborhood of the torus $\mathscr{T}$ is a family of the same type. In this setting, the family $\mathscr{F}_s^j$ is assumed to be either analytic or formal corresponding to the respective properties of the $(j + s)$-dimensional set and of the parameters h_i in it.

Theorem 2.4. *If the family $\mathscr{F}_s^j$ adjoining the torus $\mathscr{T}$ lies in the subspace $\mathscr{K}$ but in no smaller subspace and if $r' = r'(\mathscr{K}) \neq 0$, then $j = r'$ and $\mathscr{F}_s^j$ always lies in the set* Re $\mathscr{A}$. *Conversely, each connected component S of the set* Re $\mathscr{A}$ *adjoining $\mathscr{T}$ forms a family $\mathscr{F}_\sigma^{r'}$ if S lies in $\mathscr{K}$ but in no smaller subspace.*

The proof follows from the arguments of property 5 above, extended to an arbitrary subspace $\mathscr{K}$. The restriction of the theorem (i.e., $r' \neq 0$) is needed only for $k = 0$ on the integral subspace $\mathscr{L}[0]$, corresponding to the collection of all zero eigenvalues:

$$\lambda_1 = \cdots = \lambda_{l'} = 0 \qquad (\lambda_j \neq 0, j > l')$$

according to (1.15''''). On the subspace $\mathscr{L}[0]$ the set $\mathscr{A} \cap \mathscr{L}[0]$ consists of rest points (see property 6). However, in $\mathscr{L}[0]$ there may be families of periodic and conditionally periodic solutions formally adjoining $\mathscr{T}$. These families don't lie in the set $\mathscr{A}$ and they may not be analytic families of the original system (1.7).

Example 2.3. Consider the system

$$\begin{aligned} \dot{y}_1 &= 4y_2(y_1^2 + y_2^2) \\ \dot{y}_2 &= -4y_1(y_1^2 + y_2^2) \\ \dot{y}_3 &= y_3 + \varphi(y_1, y_2) \end{aligned} \tag{2.12}$$

where $k = 0, l = 2, m = 1, n = 0, \lambda_1 = \lambda_2 = 0, \mu_1 = 1$. The subsystem for y_1 and y_2 has the integral $h = (y_1^2 + y_2^2)^2$, and its solutions are periodic:

$$y_1 = \sqrt[4]{h} \cos 4\sqrt{h}t \ , \qquad y_2 = \sqrt[4]{h} \sin 4\sqrt{h}t \ . \tag{2.13}$$

The seminormalizing transformation of system (2.12) coincides in this case with the normalizing transformation and may be written as

$$y_1 = v_1 \ , \qquad y_2 = v_2 \ , \qquad y_3 = w + \xi(v_1, v_2) \ . \tag{2.14}$$

The seminormal form of (2.12) coincides with the normal form and is

$$\begin{aligned} \dot{v}_1 &= 4v_2(v_1^2 + v_2^2) \\ \dot{v}_2 &= -4v_1(v_1^2 + v_2^2) \\ \dot{w} &= w \ . \end{aligned} \tag{2.15}$$

The manifold $\mathscr{V} = \{v_1, v_2, w \colon w = 0\}$ of (2.15) is filled with the periodic solutions (2.13), and for it $r' = r(\mathscr{V}) = 0$. The set $\mathscr{A}$ is defined by the system

$$4v_2(v_1^2 + v_2^2) = -4v_1(v_1^2 + v_2^2) = 0 \ .$$

For real v_1 and v_2 this system has only the trivial solution $v_1 = v_2 = 0$. Hence, $\operatorname{Re}\mathscr{A} = \{v_1 = v_2 = w = 0\}$. But on the manifold $\operatorname{Re}\mathscr{V}$ all solutions of (2.15) are periodic and form a one-parameter family with parameter h. The formal family $\mathscr{F}_1^1$ lies in $\mathscr{V} = \mathscr{K}$ and $r'(\mathscr{K}) = 0$. Consequently, not all formal families of periodic solutions adjoining the point $v_1 = v_2 = w = 0$ belong to the set $\mathscr{A}$.

On the other hand according to Lemma 7 of § 7 [Bruno, 1971; see also example 1 of § 6] it is possible to choose a convergent series $\varphi(y_1, y_2)$ in (2.12) so that the series ξ diverges in the normalizing transformation (2.14). That is, the formal family $\operatorname{Re}\mathscr{V}$ of periodic solutions is not analytic. Thus, not all formal families of periodic solutions adjoining a fixed point are analytic.

The following is given in Bruno's paper [Bruno, 1970].

Hypothesis 1: *All formal families of periodic solutions adjoining a fixed or periodic solution $\mathscr{T}$ are analytic.*

The example considered shows that for a fixed solution $\mathscr{T}$ this hypothesis is false. It will be shown in § 4 that the hypothesis is true for all formal families $\mathscr{F}_s^1$ located outside the subspace $\mathscr{L}[0]$ for $k = 0$ (see theorem 2.4), and the hypothesis is always true for $k = 1$.

Example 2.4. Consider system (1.7) with $k = 0$, $l = 2$, m_- arbitrary, $m_+ = 0$, $n = 1$, where $m_1\lambda_1 + m_2\lambda_2 = 0$ for certain positive integral values of m_1 and m_2. That is, system (1.7) passes through a single positive resonance. Normal form (1.12) is

$$\dot{v}_j = v_j \sum_{k=0}^{\infty} g_{jk}(\varepsilon)(v_1^{m_1} v_2^{m_2})^k \equiv v_j g_j(v_1^{m_1} v_2^{m_2}, \varepsilon) \ , \qquad j = 1, 2 \ , \tag{2.15'}$$

where $g_{jk}(\varepsilon)$ are power series in ε. If we set $\tilde{v} = v_1^{m_1} v_2^{m_2}$, then

$$\dot{\tilde{v}} = \tilde{v}(m_1 g_1 + m_2 g_2) = \tilde{v} f(\tilde{v}, \varepsilon) \ .$$

The set $\mathscr{A}$ is defined by the system of equations

$$v_j g_j = \lambda_j v_j a \ , \qquad j = 1, 2 \ .$$

$\mathscr{A}$ consists of three components

$$\mathscr{A}^1 = \{V, \varepsilon: g_1 = \lambda_1 a, v_2 = 0\} = \mathscr{L}[\lambda_1] ,$$

$$\mathscr{A}^2 = \{V, \varepsilon: v_1 = 0, g_2 = \lambda_2 a\} = \mathscr{L}[\lambda_2] ,$$

$$\mathscr{A}^3 = \{V, \varepsilon: g_1 = \lambda_1 a, g_2 = \lambda_2 a\} = \{f = 0\} .$$

The components $\mathscr{A}^1$ and $\mathscr{A}^2$ are invariant coordinate subspaces. Then for $\mathscr{A}^3$ we have the equation

$$f \equiv m_1 g_1(\tilde{v}, \varepsilon) + m_2 g_2(\tilde{v}, \varepsilon) = 0 .$$

If

$$\partial f / \partial \varepsilon \neq 0 \quad \text{for} \quad \tilde{v} = \varepsilon = 0 , \tag{2.16}$$

then this equation has the unique solution $\varepsilon = \mathring{\varepsilon}(v_1^{m_1} v_2^{m_2})$. As long as $m_+ = 0$, the set $\mathscr{A}_W = \mathscr{A}_-$ is invariant. Condition α is satisfied in this case and, according to theorem 2.2, the set $\mathscr{A}_W$ is analytic. In particular the invariant component $\mathscr{A}^3_W = \mathscr{A}^3_-$ is analytic and, if (2.16) is satisfied, is also an analytic manifold. This result appears as theorem 2 in Pyartli's paper [Pyartli, 1972]. In that paper the points λ_1 and λ_2 lie on a line passing through the origin, while the remaining eigenvalues μ_i lie on one side of the line. By introducing a new time variable $\tilde{t} = \tau t$, wher τ is a suitable complex number, the line containing λ_1 and λ_2 can be made to coincide with the imaginary axis and Pyartli's case is reduced to that considered here.

If the original system is real and λ_1 and λ_2 are imaginary, then $\lambda_2 = -\lambda_1$ and the reality relation is $\bar{v}_2 = v_1$. Therefore $\tilde{v} = v_1 v_2$ is real and $\tilde{v} \geqslant 0$ for all real values of the original variables. The set $\operatorname{Re} \mathscr{A}^3$ is a one-parameter family of periodic solutions. If condition (2.16) is imposed, this family may be written in the form

$$\varepsilon = \mathring{\varepsilon}(\tilde{v}) = c_1 \tilde{v} + c_2 \tilde{v}^2 + \cdots$$

The values of ε in this expression have a fixed sign. As $t \to \infty$ the solutions in the set $\operatorname{Re} \mathscr{A}^3_-$ tend asymptotically to the family $\operatorname{Re} \mathscr{A}^3$. Here we have that

$$\operatorname{Re} \mathscr{A}^1 = \operatorname{Re} \mathscr{A}^2 = \{V, \varepsilon: v_1 = v_2 = 0\} .$$

2.4. Properties of the Set $\mathscr{B} = \mathscr{B}_W \cap \mathscr{V}$

2.4.1. The Set $\mathscr{B}^* = \mathscr{B} \cap \mathscr{V}^*$

1) First we consider the matrix B on the set $\mathscr{A}^* = \mathscr{A} \cap \mathscr{V}^*$. Since on it we have

$$\frac{\partial \Psi}{\partial V} - La = \left(v_i \frac{\partial g_i}{\partial v_j} \right) , \qquad i, j = 1, \ldots, l ,$$

then

$$B = \begin{pmatrix} \dfrac{\partial \Phi}{\partial U} & \dfrac{\partial \Phi}{\partial V} \\ \left(v_i \dfrac{\partial g_i}{\partial u_j} \right) & \left(v_i \dfrac{\partial g_i}{\partial v_j} \right) \end{pmatrix} = DB_1 D^{-1} ,$$

where D is the diagonal matrix $\{1, \dots, 1, v_1, \dots, v_l\}$ and

$$B_1 = \begin{pmatrix} \dfrac{\partial \Phi}{\partial U} & \dfrac{\partial \Phi}{\partial \ln V} \\ \dfrac{\partial G}{\partial U} & \dfrac{\partial G}{\partial \ln V} \end{pmatrix}$$

If for system (1.15″) we take an analogous matrix,

$$\tilde{B}_1 = \begin{pmatrix} \dfrac{\partial \tilde{\Phi}}{\partial U} & \dfrac{\partial \tilde{\Phi}}{\partial \ln \tilde{V}} \\ \dfrac{\partial \tilde{G}}{\partial U} & \dfrac{\partial \tilde{G}}{\partial \ln \tilde{V}} \end{pmatrix},$$

then according to (1.15′) and (1.15‴) it is similar to a matrix B_1:

$$\tilde{B}_1 = \underset{\sim}{A} B_1 \underset{\sim}{A}^{-1} ,$$

where

$$\underset{\sim}{A} = \begin{pmatrix} E & 0 \\ \underset{\sim}{N} & \underset{\sim}{K} \end{pmatrix} .$$

Thus, the matrices B_1 and $\tilde{B}_1$ are similar on the set $\mathscr{A}^*$. Hence, the set $\mathscr{B}^* = \mathscr{B} \cap \mathscr{A}^*$ is distinguished from the set $\mathscr{A}^*$ in that on $\mathscr{B}^*$ the matrix $\tilde{B}_1$ is nilpotent. Since $\tilde{\Phi}$ and $\tilde{G}$ do not depend on U, the matrix B_1 is nilpotent simultaneously with the matrix $\partial\tilde{G}/\partial(\ln \tilde{V})$. Moreover, $\tilde{G}$ is independent of $\tilde{v}_1, \dots, \tilde{v}_{l-d}$. Therefore the set $\mathscr{B}^*$ is distinguished by the nilpotency of the matrix (2.11′); i.e., on $\mathscr{B}^*$ the eigenvalues of the matrix (2.11′) all vanish (all the coefficients of the characteristic polynomial of the matrix are zero). Hence the set $\mathscr{B}^*$ is characterized within $\mathscr{A}^*$ by d equations, and the number of free parameters on the set $\mathscr{B}^*$ is

$$\sigma_B = d + n + 1 - (k + l) - d = n + 1 - (k + l) .$$

It is assumed throughout that $d \neq k + l$.

2) According to property 9 of section 2.3, solutions in the set $\operatorname{Re}\mathscr{A}^*$ are conditionally periodic, while their characteristic numbers are the eigenvalues of the matrix (2.11′) (see property 10 of section 2.3). Hence, the only tori in $\operatorname{Re}\mathscr{B}^*$ are those in $\operatorname{Re}\mathscr{A}^*$ for which all eigenvalues vanish. The set $\mathscr{B}^*$ may be described as follows: it consists of those stationary solutions of the subsystem of the last d

equations of (1.15″) for which the functions $\tilde{\Phi}, \tilde{g}_1, \ldots, \tilde{g}_{l-d}$ take on the values Ωa, $\tilde{\lambda}_1 a, \ldots, \tilde{\lambda}_{l-d} a$, and for which all eigenvalues of the subsystem vanish. The invariance of the set $\mathscr{B}^*$ in system (1.12) follows from all this.

Example 2.5. Consider the following system

$$\begin{aligned} \dot{x}_1 &= \omega + \varepsilon f(X, \varepsilon) \ , \\ \dot{x}_2 &= 1 \ , \end{aligned} \tag{2.17}$$

on a two-dimensional torus, where ε is a small parameter. Here $k = 2, l = m = 0$, $n = 1$, while the initial torus $\mathscr{T}$ has, for $\varepsilon = 0, \Omega = (\omega, 1)$, where ω is an irrational number. Condition (1.5) is satisfied and the normal form is

$$\dot{U} = \Phi(\varepsilon) = \Omega + \cdots$$

The matrix B is everywhere nilpotent since Φ does not depend on U. The set $\mathscr{A} = \mathscr{B}$ is defined by the system $\Phi(\varepsilon) = \Omega a$, which reduces to an equation $\varphi_1(\varepsilon) = \omega\varphi_2(\varepsilon)$ after dividing by a. If this equation is not identically satisfied, then near $\varepsilon = 0$ the only solution is $\varepsilon = 0$.

We introduce another small parameter δ into system (2.17):

$$\dot{x}_1 = \omega + \delta + \varepsilon f(X, \varepsilon) \ , \qquad \dot{x}_2 = 1 \ . \tag{2.18}$$

The corresponding normal form is

$$\begin{aligned} \dot{u}_1 &= \omega + \delta + \varepsilon\varphi_1^{(1)}(\varepsilon, \delta) + \delta^2 \varphi_1^{(2)}(\delta) \ , \\ \dot{u}_2 &= 1 + \varepsilon\varphi_2^{(1)}(\varepsilon, \delta) + \delta^2 \varphi_2^{(2)}(\delta) \ . \end{aligned} \tag{2.19}$$

As before, the matrix B is everywhere nilpotent. The equation for the set $\mathscr{A}$ has the following form

$$\delta + \varepsilon\varphi_1^{(1)}(\varepsilon, \delta) + \delta^2 \varphi_1^{(2)}(\delta) = \omega(\varepsilon\varphi_2^{(1)} + \delta^2 \varphi_2^{(2)}) \ .$$

Near the point $\varepsilon = \delta = 0$ it has the unique solution $\delta = \mathring{\delta}(\varepsilon)$. According to theorem 2.3 if Ω satisfies condition β, then the transformation of system (2.18) to normal form (2.19) is analytic for $\delta = \mathring{\delta}(\varepsilon)$, which is itself an analytic function. That is, for sufficiently small ε there exists $\delta = \mathring{\delta}(\varepsilon)$ such that system (2.18) is reducible to the form

$$\dot{u}_1 = \omega a \ , \qquad \dot{u}_2 = a$$

by means of an analytic transformation. In fact, this is a form of theorem 2 in § 13 of Arnold's paper [Arnold, 1961]. In order to reduce the system

$$\dot{X} = \Omega + \varepsilon \underset{\sim}{Z}(X, \varepsilon)$$

to the form

$$\dot{U} = \Omega \tag{2.19'}$$

for arbitrary $k \geqslant 2$ it is necessary to add a k-dimensional parameter Δ to the right hand part of the system. Then the normal form is

$$\dot{U} = \Omega + \Delta + O(\varepsilon) + O(|\Delta|^2) \equiv \Phi(\varepsilon, \Delta) \ .$$

The part of the set $\mathscr{A} = \mathscr{B}$ on which the parameter a is 1 is defined by the system

$$\Phi - \Omega \equiv \Delta + O(\varepsilon) + O(|\Delta|^2) = 0 \ .$$

According to the Implicit Function Theorem this system has the unique solution $\Delta = \mathring{\Delta}(\varepsilon)$, $\mathring{\Delta}(0) = 0$. According to theorem 2.3, if Ω satisfies condition β and if ε is sufficiently small, then there is a function $\mathring{\Delta}(\varepsilon)$ such that the system

$$\dot{X} = \Omega + \mathring{\Delta}(\varepsilon) + \varepsilon \underset{\sim}{Z}(X, \varepsilon)$$

may be reduced to the form (2.19′) by an analytic change of coordinates. This is theorem 9 in Arnold's paper [Arnold, 1961]. There is a misprint in the formulation of Arnold's theorem: $d\vec{u}/dt = \vec{\mu}$ should appear in the last formula, not $d\vec{u}/dt = 2\pi\vec{\mu}$.

Example 2.6. We shall now make system (2.17) more complicated by adding a third coordinate:

$$\begin{aligned} \dot{x}_1 &= \omega + \gamma y + \varepsilon f_1(X, y, \varepsilon) \ , \\ \dot{x}_2 &= 1 \ , \\ \dot{y} &= \varepsilon y f_2(X, y, \varepsilon) \ . \end{aligned} \tag{2.20}$$

Here $k = 2$, $l = 1$, $m = 0$, $n = 1$; ω is an irrational number, and the initial torus is obtained for $y = \varepsilon = 0$, and $\lambda_1 = 0$. The normal form is

$$\begin{aligned} \dot{u}_1 &= \omega + \gamma v + \varepsilon \varphi_1(v, \varepsilon) \ , \\ \dot{u}_2 &= 1 + \varepsilon \varphi_2(v, \varepsilon) \ , \\ \dot{v} &= \varepsilon v \psi(v, \varepsilon) \ . \end{aligned}$$

We assume that

$$\psi(v, \varepsilon) \equiv 0 \ ; \tag{2.21}$$

then the matrix B is everywhere nilpotent. The set $\mathscr{A} = \mathscr{B}$ is defined by the equation

$$\gamma v + \varepsilon \varphi_1 = \omega \varepsilon \varphi_2 \ ,$$

which for $\gamma \neq 0$ has a unique solution $v = \mathring{v}(\varepsilon)$ (near $v = \varepsilon = 0$). If $\Omega = (\omega, 1)$ satisfies condition β and if (2.21) is also satisfied, then according to theorem 2.3

the function $\mathring{v}(\varepsilon)$ is analytic, and for each sufficiently small ε system (2.20) has an invariant torus with a fixed frequency ratio $\omega_1(\varepsilon)/\omega_2(\varepsilon) = \omega$. If instead of a neighborhood of the torus $\mathscr{T}$ we consider the point mapping defined on the section $x_1 = 0$ by the solutions of the system, then we obtain an analytic variant of a theorem of Moser [Moser, 1962; 1973a, part 1, theorem 9]. Identity (2.21) holds either if the mapping preserves area, or if any loop intersects its image, or if the mapping is invertible, or if the divergence vanishes [Moser, 1966, § 10, example 2]. Here v plays essentially the same role as δ in the previous example even though the system is no longer defined on the torus, but in a neighborhood of the torus.

Example 2.7. In system (1.7) k and m_- are arbitrary, $l = m_+ = 0$, $n = k + 1$, and $\underset{\sim}{Z}^{(1)} = \Delta + \varepsilon\Theta(X, Y, \varepsilon)$, where $\Delta = (\delta_1, \ldots, \delta_k)$ and ε are small parameters. Then in the normal form (1.12)

$$\Phi = \Omega + \Delta + \varepsilon\Phi^{(1)}(\Delta, \varepsilon) \ ,$$

and the set $\mathscr{A}$ is defined by the system

$$\Delta + \varepsilon\Phi^{(1)}(\Delta, \varepsilon) = \Omega(a - 1) \ .$$

which has the unique solution

$$\Delta = \mathring{\Delta}(\varepsilon, a - 1) \ , \qquad \mathring{\Delta}(0, 0) = 0 \ .$$

That is, the set $\mathscr{A}$ is a manifold. Since $l = 0$ and Φ does not depend on U, the matrix B is identically zero; consequently, $\mathscr{A} = \mathscr{B}$. According to theorems 2.3 and 2.4, if Ω satisfies condition β, then the set $\operatorname{Re}\mathscr{A}$ is an analytic two-parameter family of k-dimensional tori. In the family there is a one-parameter family $\mathscr{F}_1^k$ of tori with frequencies Ω. This family is found by solving the system $\Delta + \varepsilon\Phi^{(1)} = 0$ with unique solution $\Delta = \mathring{\Delta}(\varepsilon, 0)$. Since $m_+ = 0$, the set $\mathscr{A}_W = \mathscr{A}_-$ is invariant. Solutions in $\operatorname{Re}\mathscr{A}_-$ tend asymptotically to tori in $\operatorname{Re}\mathscr{A}$ as $t \to +\infty$. The subset $\operatorname{Re}(\mathscr{F}_1^k)_W$ is also an invariant analytic manifold consisting of a one-parameter family of tori with frequencies Ω and of solutions asymptotic to these tori. The analyticity of this subset is proved in theorems 3 and 4 in Bogolyubov's lectures [Bogolyubov, 1964; see also Mitropol'skiĭ, 1964, theorem 1].

Example 2.8. Let us consider the invertible system

$$\begin{aligned} \dot{X} &= \Omega + F^{(0)}(Y) + \varepsilon F^{(1)}(X, Y, \varepsilon) \ , \\ \dot{Y} &= \varepsilon G(X, Y, \varepsilon) \ , \end{aligned} \tag{2.22}$$

where

$$F^{(1)}(-X, Y, \varepsilon) = F^{(1)}(X, Y, \varepsilon) \ ,$$

$$G(-X, Y, \varepsilon) = -G(X, Y, \varepsilon) \ ,$$

$$F^{(0)}(0) = 0 \ , \qquad Y = (y_1, \ldots, y_k) \ ,$$

and ε is a small parameter. The initial torus is $\mathscr{T} = \{X, Y, \varepsilon\colon Y = 0, \varepsilon = 0\}$; $\Lambda = 0$, and Ω is assumed to satisfy condition (1.5). The normal form is

$$\begin{aligned} \dot{U} &= \Omega + \Phi^{(0)}(V) + \varepsilon\Phi^{(1)}(V, \varepsilon) \ , \\ \dot{V} &= \varepsilon\Psi(V, \varepsilon) \ . \end{aligned} \tag{2.23}$$

It may be shown that $\Phi^{(0)}(V) = F^{(0)}(V)$ since system (2.22) is in normal form for $\varepsilon = 0$ [Bruno, 1973, corollary of material in section 2.2]. According to property 5 of section 1.4 the normal form is invertible and

$$\begin{aligned} \Phi(V, \varepsilon) &= \Phi(V, \varepsilon) \ , \\ \Psi(V, \varepsilon) &= -\Psi(V, \varepsilon) \ . \end{aligned}$$

That is, $\Psi(V, \varepsilon) \equiv 0$, while $\Phi(V, \varepsilon)$ is an arbitrary series. The matrix B is everywhere nilpotent. The set $\mathscr{A} = \mathscr{B}$ is determined by the equations

$$\Omega + F^{(0)}(V) + \varepsilon\Phi^{(1)}(V, \varepsilon) = \Omega a$$

or

$$F^{(0)}(V) + \varepsilon\Phi^{(1)}(V, \varepsilon) = \Omega(a - 1) \ .$$

If for $V = 0$ the Jacobian $\det(\partial F^{(0)}/\partial V) \neq 0$, then this system has the unique solution

$$V = \mathring{V}(\varepsilon, a - 1) \ , \qquad \mathring{V}(0, 0) = 0 \tag{2.24}$$

depending on two free parameters. If Ω satisfies condition β, then according to theorem 2.3 the family (2.24) is analytic. Here the situation is much as in example 2.7: for $a = 1$, the family (2.24) has a one-parameter family of tori with frequencies Ω. That is, the initial torus $\mathscr{T}$ can be continued in the parameter ε [Moser, 1967, § 6, section b].

2.4.2. The Set $\mathscr{B}$ on Subspaces

3) Let us consider the set $\mathscr{B}$ on the coordinate subspace (2.8′). On the set $\mathscr{A}' = \mathscr{A} \cap \mathscr{K}$ the matrix B is

$$B = \begin{pmatrix} \partial\hat{\Phi}/\partial U & \partial\hat{\Phi}/\partial V' & \partial\Phi/\partial V'' \\ \partial\hat{\Psi}'/\partial U & \partial\hat{\Psi}'/\partial V' - L'a & \partial\Psi'/\partial V'' \\ \partial\hat{\Psi}''/\partial U & \partial\hat{\Psi}''/\partial V' & \partial\Psi''/\partial V'' - L''a \end{pmatrix} \tag{2.25}$$

Here $L' = \{\lambda_1, \ldots, \lambda_{l'}\}$, and $L'' = \{\lambda_{l'+1}, \ldots, \lambda_l\}$ are diagonal matrices; the last column is calculated for $V'' = 0$, and in that column there appear the parts of the series Φ and Ψ which are linear in V''. The set $\mathscr{B}'' = \mathscr{K} \cap \mathscr{B}$ is singled out of the set $\mathscr{A}'$ by the condition $B^{k+l} = 0$.

4) If the subspace $\mathscr{K}$ is an integral subspace $\mathscr{L} = \mathscr{L}[\lambda_1, \ldots, \lambda_{l'}]$, then $\hat{\Psi}'' \equiv 0$. Hence, the matrix (2.25) is block triangular. It is nilpotent if and only if both of its diagonal blocks

$$B' = \begin{pmatrix} \dfrac{\partial \hat{\Phi}}{\partial U} & \dfrac{\partial \hat{\Phi}}{\partial V'} \\[2ex] \dfrac{\partial \hat{\Psi}'}{\partial U} & \dfrac{\partial \hat{\Psi}'}{\partial V'} - L'a \end{pmatrix}, \tag{2.26}$$

$$B'' = \frac{\partial \Psi''}{\partial V''} - L''a \tag{2.27}$$

are nilpotent. We shall take the parts of Ψ'' which are linear in V'':

$$\psi_i = \sum_{j=l'+1}^{l} c_{ij}(U, V', \underset{\sim}{E}) v_j + \cdots , \qquad i = l' + 1, \ldots, l ,$$

and form the matrix $C'' = (c_{ij})$, $i, j = l' + 1, \ldots, l$. Then $B'' = C'' - L''a$. Note that in the integral subspace $\mathscr{L} = \mathscr{L}[\lambda_1, \ldots, \lambda_{l'}]$ the normal form (1.12) induces the system

$$\begin{aligned} \dot{U} &= \hat{\Phi}(U, V', \underset{\sim}{E}) \equiv \Phi|_{V''=0} \\ \dot{V}' &= \hat{\Psi}'(U, V', \underset{\sim}{E}) \equiv \Psi'|_{V''=0} \end{aligned} \tag{2.28}$$

If $\mathscr{B}'$ denotes the set in $\mathscr{L}$ which is the set $\mathscr{B}$ for system (2.28), then

$$\mathscr{B}' = \{U, V', \underset{\sim}{E}\colon U, V', \underset{\sim}{E} \in \mathscr{A}', B'^{k+l'} = 0\} .$$

But the set $\mathscr{B}'' = \mathscr{B} \cap \mathscr{L}$ is that subset of $\mathscr{B}'$ on which the matrix B'' is nilpotent. Thus,

$$\mathscr{L} \supset \mathscr{A}' \supset \mathscr{B}' \supset \mathscr{B}'' .$$

5) Now consider the sets $\mathscr{A}'$, $\mathscr{B}'$, and $\mathscr{B}''$ outside small coordinate subspaces: Then these are the sets

$$\mathscr{A}'^* = \mathscr{A} \cap \mathscr{L}^* , \qquad \mathscr{B}'^* = \mathscr{B}' \cap \mathscr{L}^* , \qquad \mathscr{B}''^* = \mathscr{B} \cap \mathscr{L}^* .$$

$\mathscr{A}'^*$ and $\mathscr{B}'^*$ have the same meaning in $\mathscr{L}^*$ as $\mathscr{A}^*$ and $\mathscr{B}^*$ have in $\mathscr{V}^*$. The number of parameters on the set $\mathscr{B}'^*$ is

$$d' + n + 1 - (k + l') - d' = n + 1 - (k + l') .$$

The set $\mathscr{B}''^*$ is further defined by the $l - l'$ equations which are obtained by equating to zero all coefficients of the characteristic polynomial of the matrix B''. But all coefficients $\chi^{(j)}$ of the characteristic polynomial of the matrix B'' are series in V' and $\exp \mathrm{i}\langle P, U\rangle$. These series contain only those terms

$$V'^{Q'} \underset{\sim}{E}^S \exp \mathrm{i}\langle P, U\rangle ,$$

for which

$$\mathbf{i}\langle P,\Omega\rangle + \langle Q',\Lambda'\rangle = 0 \ .$$

Thus, if analogously to property 1 of section 1.4, we transform coordinates V' to coordinates $\tilde{V}'$, then the coefficients $\chi^{(j)}$ depend only on the "resonant coordinates" $\tilde{v}_{l'-d'+1}, \ldots, \tilde{v}_{l'}, \underset{\sim}{E}$. Hence the condition $B''^{l-l'} = 0$ is a restriction only on the choice of fixed points in the coordinates $\tilde{v}_{l'-d'+1}, \ldots, \tilde{v}_{l'}, \underset{\sim}{E}$. Therefore, the set $\mathscr{B}''^*$ is invariant in system (2.28). The number of free parameters in $\mathscr{B}''^*$ is

$$\sigma''_B = n + 1 - (k + l') - (l - l') = n + 1 - (k + l) \ .$$

It should be noted that the number of parameters in each component of the set $\mathscr{B}$ is unique, while the number of parameters in the components of the set $\mathscr{A}$ may diminish with the expansion of the subspace.

6) For some $i \leqslant l$, let

$$\lambda_i \neq \sum_{j \neq i} q_j \lambda_j + \mathbf{i}\langle P,\Omega\rangle \ , \tag{2.29}$$

for all integers $q_j \geqslant 0$, where $P \in \mathbb{Z}^k$. Then in the normal form (1.12) the series ψ_i does not contain terms independent of v_i, i.e., $\psi_i = v_i g_i$, where g_i is a series in non-negative powers of V. The normal form (1.12) has the integral manifold

$$\mathscr{L}[\lambda_i] = \{U, V, \underset{\sim}{E} \colon v_i = 0\} \ .$$

The set $\mathscr{A}$ splits into two components,

$$\mathscr{A}[i] = \{U, V, \underset{\sim}{E} \colon \Phi = \Omega a;\ \psi_j = \lambda_j v_j a, j \neq i, v_i = 0\} \ ,$$

$$\mathscr{A}(i) = \{U, V, \underset{\sim}{E} \colon \Phi = \Omega a;\ \psi_j = \lambda_j v_j a, j \neq i, g_i = \lambda_i a\} \ .$$

Evidently, $\mathscr{A} = \mathscr{A}[i] \cup \mathscr{A}(i)$. Now we show that

$$\mathscr{B} \subset \mathscr{A}(i) \ . \tag{2.30}$$

This need only be shown for points in $\mathscr{L}[\lambda_i]$, that is for points in $\mathscr{A}[i]$. For this the matrix B'' is the scalar

$$B'' = g_i - \lambda_i a \ .$$

It is nilpotent only if

$$g_i = \lambda_i a \ .$$

But this is the condition which distinguishes the set $\mathscr{A}(i) \cap \mathscr{L}$ from the set $\mathscr{A}[i]$. This verifies (2.30).

If inequalities (2.29) are satisfied for some i, say for $i = 1, 2, \ldots, l'$, where $l' \leqslant l$, then

$$\mathscr{B} \subset \mathscr{A}(1) \cap \mathscr{A}(2) \cap \cdots \cap \mathscr{A}(l') = \mathscr{A}(1,, 2, \ldots, l') \ , \tag{2.31}$$

where

$$\mathscr{A}(1,\ldots,l') = \{U, V, \underset{\sim}{E} \colon \Phi = \Omega a;\ g_i = \lambda_i a, i = 1,\ldots,l';\ \psi_j = \lambda_j a v_j, j = l' + 1,\ldots,l\} \ .$$

§ 3. A Hamiltonian System

3.1. Statement of the Problem

We shall consider the case where system (1.1) is a Hamiltonian system with s degrees of freedom:

$$\dot{z}_i = \frac{\partial f}{\partial z_{s+i}} , \qquad \dot{z}_{s+i} = -\frac{\partial f}{\partial z_i} , \qquad i = 1, \ldots, s , \tag{3.1}$$

where $f = f(Z, \underset{\sim}{E})$, $Z = (z_1, \ldots, z_{2s})$, i.e., $k + l + m = 2s$. Let system (3.1) have a k-dimensional initial torus $\mathscr{T}$. Then it is possible to introduce local canonical coordinates X, Y so that system (1.7) is the Hamiltonian system

$$\begin{aligned} \dot{x}_i &= \frac{\partial f}{\partial y_i} , \qquad \dot{y}_i = -\frac{\partial f}{\partial x_i} , \qquad i = 1, \ldots, k , \\ \dot{y}_{k+j} &= \frac{\partial f}{\partial y_{s+j}} , \qquad \dot{y}_{s+j} = -\frac{\partial f}{\partial y_{k+j}} , \qquad j = 1, \ldots, s - k , \end{aligned} \tag{3.2}$$

where $f = f(X, Y, \underset{\sim}{E})$. If $Y = 0$ and $\underset{\sim}{E} = 0$, then

$$\frac{\partial f}{\partial y_i} = \omega_i , \qquad \frac{\partial f}{\partial x_i} = \frac{\partial f}{\partial y_{k+j}} = \frac{\partial f}{\partial y_{s+j}} = 0 .$$

Moreover, for $\underset{\sim}{E} = 0$ the constant, linear, and quadratic terms (with respect to Y) of the series for f do not depend on X. The matrix A of the linear part of system (3.2) has k zero eigenvalues

$$\lambda_1 = \cdots = \lambda_k = 0 ; \tag{3.3}$$

moreover, $\tilde{l}$ pairs lie on the imaginary axis

$$\lambda_{k+j} = -\lambda_{k+\tilde{l}+j} , \qquad \operatorname{Re} \lambda_{k+j} = 0 , \qquad j = 1, \ldots, \tilde{l} , \tag{3.4}$$

and there are $m_- = m_+$ pairs of complex eigenvalues

$$\mu_j = -\mu_{m_-+j}\,, \qquad \operatorname{Re}\mu_j < 0\,, \qquad j = 1, \ldots, m_- \tag{3.5}$$

Thus,

$$l = k + 2\tilde{l}\,, \qquad m = 2m_- = 2m_+\,,$$

$$2(k + \tilde{l} + m_-) = k + l + m = 2s\,.$$

First let us consider the case $m = 0$ when all eigenvalues have zero real parts and $k + \tilde{l} = s$. Then the normalizing transformation can be assumed to be canonical and the normal form (1.12) is the Hamiltonian system

$$\begin{gathered} \dot{u}_i = \frac{\partial h}{\partial v_i}\,, \qquad \dot{v}_i = -\frac{\partial h}{\partial u_i}\,, \qquad i = 1, \ldots, k\,, \\ \dot{v}_{k+j} = \frac{\partial h}{\partial v_{k+\tilde{l}+j}}\,, \qquad \dot{v}_{k+\tilde{l}+j} = -\frac{\partial h}{\partial v_{k+j}}\,, \qquad j = 1, \ldots, \tilde{l}\,, \end{gathered} \tag{3.6}$$

where

$$h = \sum h_{PQS} V^Q \underset{\sim}{E}^S \exp \mathbf{i}\langle P, U\rangle \tag{3.7}$$

and the summation is carried out only for those $P \in \mathbb{Z}^k$, $0 \leqslant Q \in \mathbb{Z}^l$, $0 \leqslant S \in \mathbb{Z}^n$ which satisfy equation (1.15) [Bruno, 1971, theorem 12 in § 12; 1970, theorem 1]. Thanks to property (3.3), the variables $v_1, \ldots, v_k$ and the parameter $\underset{\sim}{E}$ appear in the series expansion of (3.7) with non-negative integral exponents. Equation (1.15) only imposes bounds on the exponents of the variables $v_{k+1}, \ldots, v_{k+2\tilde{l}}$.

Example 3.1. Let $k = l = s$, $m = 0$. Then $\Lambda = 0$ and, thanks to condition (1.5), the solutions of equations (1.15) are $P = 0$, Q arbitrary. Thus, in normal form (3.6), the Hamiltonian (3.7) depends only on $V = (v_1, \ldots, v_k)$. According to (3.6), we have in the notation of (1.12) that

$$\Phi = \frac{\partial h}{\partial V} = \Omega + \cdots\,, \qquad \Psi = -\frac{\partial h}{\partial U} \equiv 0\,.$$

The equations for the set $\mathscr{A}$ are

$$\frac{\partial h}{\partial V} = \Omega a\,. \tag{3.8}$$

The matrix B is always nilpotent since

$$B = \begin{pmatrix} h_{VU} & h_{VV} \\ -h_{UU} & -h_{UV} \end{pmatrix} = \begin{pmatrix} 0 & h_{VV} \\ 0 & 0 \end{pmatrix}$$

Let $n = 0$ and $h = \langle \Omega, V\rangle + \frac{1}{2}\langle V, CV\rangle + O(|V|^3)$, where C is a symmetric matrix. Then $\partial h/\partial V = \Omega + CV + \cdots$, and equation (3.8) has the form

$$CV = \Omega(a - 1) + \cdots$$

If

$$\det C \neq 0 \ , \tag{3.9}$$

then according to the Implicit Function Theorem this equation has the unique solution

$$V = \mathring{V}(a-1) = C^{-1}\Omega(a-1) + O((a-1)^2) \ ,$$

where $a-1$ may be considered to be a small parameter. In the general case where condition β and (3.9) are satisfied, then according to theorems 2.3 and 2.4 the k-dimensional torus $\mathscr{T}$ is located in an analytic one-parameter family of k-dimensional tori; their frequencies Ωa are proportional to the set of frequencies Ω of the initial torus.

Example 3.2. Now suppose that in the previous example there is a small parameter ε (i.e., suppose that $n = 1$):

$$h = \langle \Omega, V \rangle + \tfrac{1}{2}\langle V, CV \rangle + \varepsilon\langle \Pi, V \rangle + O((|V| + |\varepsilon|)^3) \ , \qquad \Pi = \text{const} \ ;$$

then $\dfrac{\partial h}{\partial V} = \Omega + CV + \varepsilon\Pi + \cdots$. Consider equation (3.8) for $a = 1$:

$$\Omega + CV + \varepsilon\Pi + \cdots = \Omega \ ,$$

i.e.,

$$CV + \varepsilon\Pi + \cdots = 0 \ .$$

Condition (3.9) implies that this equation has a unique solution

$$V = \mathring{V}(\varepsilon) = -\varepsilon C^{-1}\Pi + O(\varepsilon^2) \ .$$

It follows from theorems 2.3 and 2.4 that, where conditions β and (3.9) are imposed on the Hamiltonian system with k degrees of freedom and one small parameter, the k-dimensional torus $\mathscr{T}$ lies in an analytic one-parameter family of invariant tori with the same set of frequencies Ω. This result is theorem 1 in a paper of Kolmogorov [Kolmogorov, 1954]. Conditions (3) and (4) given in that paper guarantee that our conditions β and (3.9) are met. In order to obtain the coordinate P used in Kolmogorov's paper, it is necessary to set $P = V - \mathring{V}(\varepsilon)$; then the family is defined by $P = 0$. For $k = 2$ the corresponding theorem was proved by Barrar[Barrar, 1966], but its formulation is needlessly complicated. Indeed here there is a two-parameter (ε and a) family of k-dimensional tori with frequencies Ωa, and in this family there is a one-parameter family with frequencies Ω. Actually, system (3.8) for arbitrary a is

$$CV + \varepsilon\Pi + \cdots = \Omega(a-1) \ . \tag{3.10}$$

It has a unique two-dimensional solution if the $k \times (k+1)$ matrix

$$(C\Omega) \tag{3.11}$$

has at least one minor of dimension k which is not zero. We shall now consider the constant-energy continuation of the initial torus $\mathscr{T}$, that is, the continuation along the level sets of constant energy, $h =$ const. To solve this problem it is necessary to add to system (3.10) the condition

$$h \equiv \langle V, \Omega \rangle + \cdots = 0 \ .$$

All systems have a unique one-dimensional solution $V = \mathring{V}(\varepsilon)$, $a = \mathring{a}(\varepsilon)$, if

$$\det \begin{pmatrix} C & \Omega \\ \Omega' & 0 \end{pmatrix} \neq 0 \ .$$

i.e., if at $V = 0$ and $\varepsilon = 0$

$$\det \begin{pmatrix} h_{VV} & h_V \\ h_V & 0 \end{pmatrix} \neq 0 \qquad (3.12)$$

The last condition was suggested by Arnold [Arnold, 1963, footnote on page 19], instead of Kolmogorov's condition (3.9). In fact, it is enough to require that the matrix (3.11) be nonsingular. This condition is weaker than (3.9) and (3.12).

If $m > 0$, then by the formal canonical change of coordinates, system (3.2) is carried into the Hamiltonian seminormal form:

$$\dot{u}_i = \frac{\partial f}{\partial v_i} \ , \qquad \dot{v}_i = -\frac{\partial f}{\partial u_i} \ , \qquad i = 1, \ldots, k \ ,$$

$$\dot{v}_{k+j} = \frac{\partial f}{\partial v_{k+\tilde{l}+j}} \ , \qquad \dot{v}_{k+\tilde{l}+j} = -\frac{\partial f}{\partial v_{k+j}} \ , \qquad j = 1, \ldots, \tilde{l} \ ; \qquad (3.13)$$

$$\dot{w}_j = \frac{\partial f}{\partial w_{m_-+j}} \ , \qquad \dot{w}_{m_-+j} = -\frac{\partial f}{\partial w_j} \ , \qquad j = 1, \ldots, m_- \ ,$$

where the Hamiltonian

$$f = h(U, V, \underset{\sim}{E}) + \check{h}(U, V, W, \underset{\sim}{E}) + \check{f}(U, V, W, \underset{\sim}{E}) \qquad (3.14)$$

has the following properties:

1. The h term in the series for f does not depend on W, and the expansion (3.7) for h has the same properties as the normal form (3.6).

2. The term $\check{h}$ contains all the terms in f which are linear in at least one of the two groups of coordinates,

$$W_- = (w_1, \ldots, w_{m_-}) \text{ or } W_+ = (w_{m_-+1}, \ldots, w_{2m_-}) \ .$$

The expansion

$$\check{h} = \sum \check{h}_{PQRS} V^Q W^R \underset{\sim}{E}^S \exp \mathrm{i} \langle P, U \rangle$$

contains only those terms for which $\langle R, \operatorname{Re} \underset{\sim}{M} \rangle = 0$.

3. $\check{f}$ contains all terms of f whose degrees in W_- or W_+ are greater than 1.

It is not hard to show that, for the Hamiltonian (3.14), system (3.13) is a seminormal form in the sense of section 1.5. The system has formal integral manifolds $\mathscr{V}$, $\mathscr{W}_-$, $\mathscr{W}_+$, $\mathscr{W}_j$, but the coordinates W do not affect the sets $\mathscr{A}_W$ and $\mathscr{B}_W$ which are defined only by the term h in (3.14).

Example 3.3. Let $\tilde{l} = 0$, k and m arbitrary. According to examples 3.1 and 3.2, the real part of the set $\mathscr{A}_W = \mathscr{B}_W$ contains an $(n+1)$-parameter family, $\operatorname{Re}\mathscr{A}$, of k-dimensional tori. Together with each torus $\mathscr{T}^0 \subset \operatorname{Re}\mathscr{A}$, the set $\operatorname{Re}\mathscr{A}_W$ also contains the s-dimensional "tendrils" $\mathscr{T}^0_-$ and $\mathscr{T}^0_+$, which tend asymptotically to the torus $\mathscr{T}^0$ and lie in the invariant sets $\operatorname{Re}\mathscr{A}_-$ and $\operatorname{Re}\mathscr{A}_+$. If Kolmogorov's condition (3.9) is satisfied and if ε is a single small parameter, then there is a one-parameter real family of "tendrilled" tori with the same frequencies Ω as the initial torus. According to theorem 2.3, condition β implies the family is analytic for this system. This result is the basic theorem in Graff's dissertation [Graff, 1971, 1974; see also Moser, 1973b, theorem 3]. Theorem 3 in Bibikov's paper [Bibikov, 1973a] asserts only the existence of k-dimensional tori with frequencies Ω and does not consider the "tendrils". A particular case of this theorem appears in a paper of Moser [Moser, 1967, theorem 7], where there is an additional assumption that every μ_j is real and the matrix A is diagonable.

Thus, for a Hamiltonian system with s degrees of freedom, a torus of arbitrary dimension k is structurally stable if k of its eigenvalues vanish and the remaining do not lie on the imaginary axis. Moreover in a system free of parameters these tori form a one-parameter family.

There are similar results for invertible systems; see example 2.8 and Bibikov's work [Bibikov, 1970a].

3.2. The Non-resonant Case

Let

$$\mathrm{i}\langle P, \Omega\rangle + q_{k+1}\lambda_{k+1} + \cdots + q_{k+\tilde{l}}\lambda_{k+\tilde{l}} \neq 0 \tag{3.15}$$

for all nonzero integral P and q_j. Thanks to (3.3) and (3.4) all solutions P, Q of equation (1.15) are such that $P = 0$, $q_1, \ldots, q_k$ are arbitrary, $q_{k+j} = q_{k+\tilde{l}+j}$ for $j = 1, \ldots, \tilde{l}$. Set

$$\rho_{k+j} = v_{k+j} v_{k+\tilde{l}+j}\,, \qquad j = 1, \ldots, \tilde{l}\,. \tag{3.15'}$$

If condition (3.15) is imposed, then the Hamiltonian (3.7) depends only on $v_1, \ldots, v_k$, $\rho_{k+1}, \ldots, \rho_{k+\tilde{l}}$. Therefore, the normal form (3.6) has the form

$$\begin{gathered}\dot{u}_i = \frac{\partial h}{\partial v_i}\,, \qquad \dot{v}_i = -\frac{\partial h}{\partial u_i} \equiv 0\,, \qquad i = 1, \ldots, k\,, \\ \dot{v}_{k+j} = v_{k+j}\frac{\partial h}{\partial \rho_{k+j}}\,, \qquad \dot{v}_{k+\tilde{l}+j} = -v_{k+\tilde{l}+j}\frac{\partial h}{\partial \rho_{k+j}}\,, \qquad j = 1, \ldots, \tilde{l}\,.\end{gathered} \tag{3.16}$$

According to (3.15) inequality (2.29) is satisfied for all $i = k + 1, \ldots, k + 2\tilde{l}$. Thus, according to (2.31),

$$\begin{aligned} \mathscr{B} &\subset \mathscr{A}(k+1,\ldots,k+2\tilde{l}) \\ &= \left\{U, V, \underset{\sim}{E}: \frac{\partial h}{\partial v_i} = \omega_i a, i = 1,\ldots,k; \frac{\partial h}{\partial \rho_j} = \lambda_j a, j = k+1,\ldots,k+\tilde{l}\right\} . \end{aligned} \tag{3.17}$$

Since the Hamiltonian h does not depend on U, then, if system (3.16) is written in the form (1.12), we have $\psi_1 \equiv \cdots \equiv \psi_k \equiv 0, \partial\Phi/\partial U \equiv 0$ and $\partial\Psi/\partial U \equiv 0$. Hence the matrix (2.4) is nilpotent together with the matrix

$$B_0 = \left(\frac{\partial \psi_i}{\partial v_j} - \delta_{ij}\lambda_j v_j a\right) , \qquad i, j = k+1, \ldots, k+2\tilde{l} ,$$

where δ_{ij} is the Kronecker symbol. According to (3.16) and (3.17) the matrix B_0 on the set $\mathscr{A}(k+1,\ldots,k+2\tilde{l})$ is the block matrix

$$B_0 = \begin{pmatrix} D_1 H D_2 & D_1 H D_1 \\ -D_2 H D_2 & -D_2 H D_1 \end{pmatrix} \tag{3.18}$$

where

$$H = \left(\frac{\partial^2 h}{\partial \rho_i \partial \rho_j}\right) , \qquad i, j = k+1, \ldots, k+\tilde{l} ,$$

$$D_1 = \{v_{k+1}, \ldots, v_{k+\tilde{l}}\} , \qquad D_2 = \{v_{k+\tilde{l}+1}, \ldots, v_{k+2\tilde{l}}\} ,$$

(D_1 and D_2 are diagonal matrices). It is not hard to see that the square of matrix (3.18) is the zero matrix since $D_1 D_2 = D_2 D_1$. Hence, according to (3.17),

$$\mathscr{B} = \mathscr{A}(k+1,\ldots,k+2\tilde{l}) . \tag{3.19}$$

First let us consider the properties of the complex set $\mathscr{B}$. The number of variables $v_1, \ldots, v_k, \rho_{k+1}, \ldots, \rho_{k+\tilde{l}}, \underset{\sim}{E}, a$ is $k + \tilde{l} + n + 1$. According to (3.17), the number of equations defining the set (3.19) is $k + \tilde{l}$. Thus, the number of free parameters in the set (3.19) is

$$k + \tilde{l} + n + 1 - (k + \tilde{l}) = n + 1 .$$

Now let

$$\begin{aligned} \check{\rho}_i &= v_i , \qquad i = 1, \ldots, k ; \\ \check{\rho}_{k+j} &= -\mathbf{i}\rho_{k+j} , \qquad j = 1, \ldots, \tilde{l} ; \\ \check{P} &= (\check{\rho}_1, \ldots, \check{\rho}_k, \check{\rho}_{k+1}, \ldots, \check{\rho}_{k+\tilde{l}}) ; \\ \check{\Omega} &= (\omega_1, \ldots, \omega_k, \mathbf{i}\lambda_{k+1}, \ldots, \mathbf{i}\lambda_{k+\tilde{l}}) . \end{aligned} \tag{3.20}$$

Then

$$h = h^0(\underset{\sim}{E}) + \langle \check{P}, \check{\Omega}\rangle + \tfrac{1}{2}\langle \check{P}, C\check{P}\rangle + \langle \check{P}, \Pi \underset{\sim}{E}\rangle + O((|\check{P}| + |\underset{\sim}{E}|)^3) , \tag{3.21}$$

where C is a square symmetric matrix of size $k + \tilde{l}$, Π is a rectangular matrix of size $(k + \tilde{l}) \times n$. According to (3.17) the set (3.19) is defined by the system of equations

$$\partial h/\partial \breve{P} = \breve{\Omega} a \ , \tag{3.22}$$

which according to (3.21) has the form

$$\breve{\Omega} + C\breve{P} + \Pi \underset{\sim}{E} + \cdots = \breve{\Omega} a \ ,$$

i.e.,

$$C\breve{P} + \Pi \underset{\sim}{E} + \cdots = \breve{\Omega}(a - 1) \ . \tag{3.23}$$

We shall find conditions under which the solutions of system (3.23) form a manifold in the coordinates $\breve{P}$ and $\underset{\sim}{E}$, i.e., system (3.23) can be solved for the parameter $a - 1$ and for $k + \tilde{l} - 1$ of the coordinates in $\breve{P}$, $\underset{\sim}{E}$. The following suffices:

$$\operatorname{rank}(C\Pi\breve{\Omega}) = k + \tilde{l} \ . \tag{3.24}$$

In fact, as long as $\breve{\Omega} \neq 0$, in the presence of condition (3.24), the matrix $(C\Pi\breve{\Omega})$ always has a minor of size $k + \tilde{l}$ which is not zero and contains the column $\breve{\Omega}$. Using this minor in the Implicit Function Theorem, we obtain the required solution of system (3.23). According to (3.15′) and (3.20) the coordinates $\rho_{k+1}, \ldots, \rho_{k+\tilde{l}}$ are expressed nonlinearly in local coordinates V. Therefore, the set (3.19) is a manifold in the local coordinates V if the system of equations (3.23) can be solved for $a - 1$ and for $k + \tilde{l} - 1$ of the variables $\breve{\rho}_1, \ldots, \breve{\rho}_k$, $\underset{\sim}{E}$, in such a way that the coordinates $\breve{\rho}_{k+1}, \ldots, \breve{\rho}_{k+\tilde{l}}$ are free parameters. A sufficient condition for this to occur is that

$$\operatorname{rank}(\tilde{C}\Pi\breve{\Omega}) = k + \tilde{l} \ , \tag{3.25}$$

where the size of the matrix $\tilde{C}$ is $(k + \tilde{l}) \times k$; $\tilde{C}$ consists of the first k columns of C. This is only possible if $\tilde{l} \leqslant n + 1$. Finally, the set (3.19) is a manifold in the coordinates V for all small values of the parameters $\underset{\sim}{E}$ if

$$\operatorname{rank}(\tilde{C}\breve{\Omega}) = k + \tilde{l} \ , \tag{3.26}$$

which is possible only for $\tilde{l} \leqslant 1$.

Now let us consider the properties of the real part of the set (3.19). The corresponding reality relations for V are

$$\bar{v}_i = v_i \ , \qquad i = 1, \ldots, k \ ,$$

$$\bar{v}_{k+j} = -\mathbf{i} v_{k+\tilde{l}+j} \ , \qquad j = 1, \ldots, \tilde{l} \ .$$

See, for example, Siegel's work [Siegel, 1956, the end of § 13]. Hence, v_i is any real number, and

$$-\arg v_{k+j} = -\frac{\pi}{2} + \arg v_{k+\tilde{l}+j} \ ,$$

i.e., according to (3.15′)

$$\arg \rho_{k+j} = \arg v_{k+j} + \arg v_{k+\tilde{l}+j} = \frac{\pi}{2}$$

and

$$\bar{\rho}_{k+j} = -\rho_{k+j} \ .$$

This means that ρ_{k+j} has pure imaginary values and $\operatorname{Im} \rho_{k+j} \geqslant 0$. Thus, according to (3.20) the reality relations for $\check{P}$ are

$$\operatorname{Im} \check{P} = 0 \ , \qquad \check{\rho}_{k+1} \geqslant 0, \dots, \check{\rho}_{k+\tilde{l}} \geqslant 0 \ . \tag{3.27}$$

Note that, for the solution of the normal form (3.16), $\check{P} = \check{P}^0 = \text{const}$. The torus $\mathscr{T}^0 \subset \operatorname{Re} \mathscr{B}$ has $k + j$ frequencies if $l - j$ of the numbers $\check{\rho}^0_{k+1}, \dots, \check{\rho}^0_{k+\tilde{l}}$ are zero. The vector $\check{\Omega}$ and the matrix C in (3.21) are real if the Hamiltonian h is real.

Now let us consider the situation for several different values of $\tilde{l}$. The case $\tilde{l} = 0$ was taken up in examples 3.1, 3.2, and 3.3.

$\tilde{l} = 1$. Suppose (3.26) is satisfied, i.e.,

$$\det(\tilde{C}\check{\Omega}) \neq 0 \ . \tag{3.28}$$

Then system (3.23) has the unique solution

$$a = \mathring{a}(\check{\rho}_{k+1}, \underset{\sim}{E}) \ , \qquad \check{\rho}_i = \mathring{\rho}_i(\check{\rho}_{k+1}, \underset{\sim}{E}) \ , \qquad i = 1, \dots, k \ . \tag{3.29}$$

Restrictions (3.27) are reduced to the single restriction, $\check{\rho}_{k+1} \geqslant 0$. If $\check{\rho}_{k+1} = 0$ then (3.29) gives k-dimensional tori in the set $\operatorname{Re} \mathscr{B}$, while for $\check{\rho}_{k+1} > 0$ there are $(k + 1)$-dimensional tori in the set $\operatorname{Re} \mathscr{B}$. Thus, the set $\operatorname{Re} \mathscr{B}$ consists of two families $\mathscr{F}^k_n$ and $\mathscr{F}^{k+1}_{n+1}$, where $\mathscr{F}^k_n$ is the boundary of $\mathscr{F}^{k+1}_{n+1}$.

If $n = 0$, then the family $\mathscr{F}^k_n = \mathscr{F}^k_0$ consists solely of the initial torus $\mathscr{T}$, while the family $\mathscr{F}^{k+1}_{n+1} = \mathscr{F}^{k+1}_1$ is the one-parameter family of $(k + 1)$-dimensional tori emanating from the initial torus $\mathscr{T}$. That is, for a system in general position the k-dimensional torus with a single pair of pure imaginary eigenvalues is the boundary of an analytic family of $(k + 1)$-dimensional tori all of whose eigenvalues have nonzero real parts.

If $n = 1$, then $\mathscr{F}^k_n = \mathscr{F}^k_1$ and $\mathscr{F}^{k+1}_{n+1} = \mathscr{F}^{k+1}_2$. For each value of the single parameter ε there is a family of $(k + 1)$-dimensional tori terminating at a k-dimensional torus. In figure 1 there is a representation of the projection of the set $\mathscr{B}$ on the plane $\check{\rho}_k, \check{\rho}_{k+1}$ for various values of ε. The heavy lines denote the real parts of the projections of the set $\operatorname{Re} \mathscr{B}$, while the marked points correspond to k-dimensional tori. Thus, if the initial torus $\mathscr{T}$ has only one pair of pure imaginary eigenvalues, and if conditions β, (3.15) and (3.28) hold, then, according to theorem 2.3, near the torus $\mathscr{T}$ there is an analytic family $\mathscr{F}^{k+1}_2$ bounded by the analytic family $\mathscr{F}^k_1$. The existence of the family $\mathscr{F}^k_1$ under the additional condition $m_- = 0$ was shown by Moser [Moser, 1967, theorem 6]. Moser's condition that the Jacobian of the vector (6.15) for $V = 0$ not be zero is equivalent

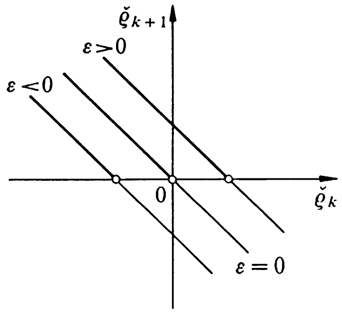

Fig. 1

to our condition (3.28). In fact Moser's vector (6.15) is made up of the components

$$\frac{a_i}{\alpha} = \frac{1}{\alpha}\frac{\partial h}{\partial v_i} , \qquad i = 1, \ldots, k , \tag{3.30}$$

where $\alpha = \mathrm{i}h_{v_{k+1}v_{k+2}} = \mathrm{i}\lambda_1 + \cdots$. The Jacobian of the vector (3.30) is

$$\begin{aligned} \det\left(\partial\left(\frac{a_i}{\alpha}\right)\Big/\partial v_j\right) &= \det\left(\frac{1}{\alpha}h_{v_iv_j} - \frac{1}{\alpha^2}h_{v_i}\alpha_{v_j}\right) \\ &= \frac{1}{\alpha^{k+1}}\det\begin{pmatrix} h_{v_iv_j} & h_{v_i} \\ \mathrm{i}h_{v_jv_{k+1}v_{k+2}} & \mathrm{i}h_{v_{k+1}v_{k+2}} \end{pmatrix} , \qquad i, j = 1, \ldots, k . \end{aligned} \tag{3.31}$$

Now note that for $V = 0$ the last matrix is the matrix $(\tilde{C}\check{\Omega})$ of (3.28). On the other hand if we set $\Delta = \alpha^2$, then $\Delta_{v_j} = 2\alpha\alpha_{v_j}$ and $\frac{1}{\alpha}\alpha_{v_j} = \frac{1}{2\Delta}\Delta_{v_j}$. Thus, the second determinant in (3.31) can be written in the form

$$\det\left(\frac{1}{\alpha}h_{v_iv_j} - \frac{1}{\alpha}\frac{1}{2\Delta}h_{v_i}\Delta_{v_j}\right) = \frac{1}{2\Delta\alpha^k}\det\begin{pmatrix} h_{v_iv_j} & h_{v_i} \\ \Delta_{v_j} & 2\Delta \end{pmatrix} \qquad i, j = 1, \ldots, k . \tag{3.32}$$

As long as $\alpha = \mathrm{i}\lambda_1 \neq 0$ for $V = 0$, then the determinants in (3.28), (3.31) and (3.32) vanish simultaneously. Note that Moser's condition (6.16) has a misprint: the factor 2 in front of Δ in (3.32) is missing. Moser used some weaker arithmetical conditions on Ω and Λ than β and (3.15) [Moser, 1967, theorem 6] since he was looking only for the family $\mathscr{F}_1^k$ and not for the family $\mathscr{F}_2^{k+1}$. In § 5 it is shown how the arithmetical conditions may be weakened for families located in coordinate subspaces (in the given case, on the subspace $v_{k+1} = v_{k+2} = 0$).

$\underline{\tilde{l} = 2.}$ Restrictions for coordinates (3.27) are given by

$$\check{\rho}_{k+1} \geqslant 0 , \qquad \check{\rho}_{k+2} \geqslant 0 . \tag{3.33}$$

First we consider the case $n = 0$. Since $\tilde{l} = 2 > n + 1 = 1$, then condition (3.25) is not satisfied. Supposed (3.24) is satisfied, where to be specific we assume that the matrix $(\check{\Omega}C)$ has its extreme left-hand minor different from zero. Then the solution of system (3.23) has the form

$$a = \mathring{a}(\check{\rho}_{k+2}) = 1 + \gamma_0 \check{\rho}_{k+2} + \cdots ,$$
$$\check{\rho}_i = \mathring{\rho}_i(\check{\rho}_{k+2}) = \gamma_i \check{\rho}_{k+2} + \cdots , \qquad i = 1, \ldots, k + 1 . \tag{3.34}$$

We shall consider two fundamental cases:

1) $\gamma_{k+1} < 0$; then $\check{\rho}_{k+1}$ and $\check{\rho}_{k+2}$ in the solution (3.34) have different signs;
2) $\gamma_{k+1} > 0$; then $\check{\rho}_{k+1}$ and $\check{\rho}_{k+2}$ in the solution (3.34) have the same signs.

In the first case the set $\operatorname{Re}\mathscr{B}$ consists of a single initial torus $\mathscr{T}$ since only on $\mathscr{T}$ are both inequalities (3.33) for the solutions (3.34) satisfied. In the second case the set $\mathscr{B}$ contains the component $\mathscr{F}_1^{k+2}$ with $\check{\rho}_{k+1} > 0$ and $\check{\rho}_{k+2} > 0$. The initial k-dimensional torus $\mathscr{T}$ bounds the family $\mathscr{T}_1^{k+2}$.

If $n = 1$, then for fixed ε the set $\mathscr{B}$ has a single parameter. As before, let $\operatorname{rank}(\check{\Omega}C) = k + \tilde{l}$. If for $\varepsilon = 0$ we have case 1, then the construction of the set $\operatorname{Re}\mathscr{B}$ depends upon the sign of ε. For one sign (say, $\varepsilon < 0$) the set $\operatorname{Re}\mathscr{B}$ is empty; for the other sign ($\varepsilon > 0$) the set $\operatorname{Re}\mathscr{B}$ consists of the family $\mathscr{F}_1^{k+2}$ bounded by two $(k + 1)$-dimensional tori. See figure 2 for a representation of the projection of $\mathscr{B}$ with ε fixed on the plane $\check{\rho}_{k+1}, \check{\rho}_{k+2}$. The heavy lines denote the real part of $\mathscr{B}$ (the family $\mathscr{F}_1^{k+2}$) while the marked points denote the k and $(k + 1)$-dimensional tori. If for $\varepsilon = 0$ we have case 2, then for any fixed $\varepsilon \neq 0$ we have a family $\mathscr{F}_1^{k+2}$ bounded by a single $(k + 1)$-dimensional torus (see figure 3). In both cases the $(k + 1)$-dimensional tori form two one-parameter families (with parameter ε) merging with and vanishing into the initial torus $\mathscr{T}$.

$\underline{\tilde{l} > 2.}$ We shall consider only the case where $n + 1 \geqslant \tilde{l}$ and (3.25) holds. To be specific, we assume that the left minor of the matrix $(\check{\Omega}\check{C}\Pi)$ is different from

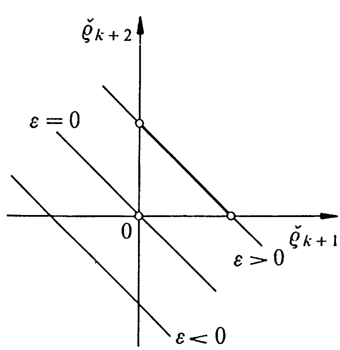

Fig. 2

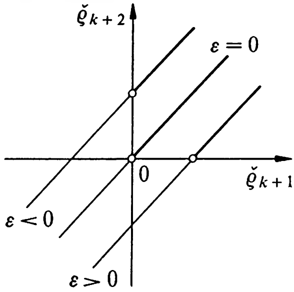

Fig. 3

zero. Then system (3.23) has a unique solution of the form

$$\begin{aligned} a &= \mathring{a}(\breve{\rho}_{k+1}, \ldots, \breve{\rho}_{k+\tilde{l}}, \varepsilon_{\tilde{l}}, \ldots, \varepsilon_n) \ , \\ \breve{\rho}_i &= \mathring{\rho}_i(\breve{\rho}_{k+1}, \ldots, \breve{\rho}_{k+\tilde{l}}, \varepsilon_{\tilde{l}}, \ldots, \varepsilon_n) \ , \qquad i = 1, \ldots, k \ , \\ \varepsilon_j &= \mathring{\varepsilon}_j(\breve{\rho}_{k+1}, \ldots, \breve{\rho}_{k+\tilde{l}}, \varepsilon_{\tilde{l}}, \ldots, \varepsilon_n) \ , \qquad j = 1, \ldots, \tilde{l} - 1 \ . \end{aligned} \tag{3.35}$$

The real part Re$\mathscr{B}$ is defined by conditions (3.27). If among the numbers $\breve{\rho}_{k+1}$, $\ldots$, $\breve{\rho}_{k+\tilde{l}}$, r of them are set equal to 0 ($0 \leqslant r \leqslant \tilde{l}$) and the remaining are variable, but positive, then the solution (3.35) gives a family $\mathscr{F}^{k+\tilde{l}-r}_{n+1-r}$. For each r we have

$$C^r_{\tilde{l}} = \frac{\tilde{l}!}{r!(\tilde{l} - r)!}$$

different families of this kind. For $r = 0$ we obtain the unique family $\mathscr{F}^{k+\tilde{l}}_{n+1}$. If $\tilde{l} = n + 1$ then the unique family with $r = \tilde{l}$ is $\mathscr{F}^k_0$, which consists of the single torus $\mathscr{T}$. If $\tilde{l} \leqslant n$, then the family with $r = \tilde{l}$ is $\mathscr{F}^k_{n+1-\tilde{l}}$ and contains k-dimensional tori for all sufficiently small $\varepsilon_{\tilde{l}}, \ldots, \varepsilon_n$.

Theorem 3.1. *If conditions β, (3.15) and (3.25) are satisfied in system (3.2) with $\tilde{l} \leqslant n + 1$, then according to theorem 2.3 there are the analytic families $\mathscr{F}^{k+\tilde{l}-r}_{n+1-r}$, $r = 0, \ldots, \tilde{l}$ which adjoin the initial torus $\mathscr{T}$. For sufficiently small $\varepsilon_{\tilde{l}}, \ldots, \varepsilon_n$ and arbitrarily small $\delta > 0$ there are tori with $k + 1, \ldots, k + \tilde{l}$ distinct frequencies in the region $|Y| < \delta$, $|\varepsilon_1| + \cdots + |\varepsilon_{\tilde{l}-1}| < \delta$. If $\tilde{l} \leqslant n$, then there is also a k-dimensional torus.*

Krasinskiĭ considered the case $n = \tilde{l} + 1$ [Krasinskiĭ, 1969], the smallness of the disturbance g in the Hamiltonian (2) playing the role of $\varepsilon_{\tilde{l}+1}$, while $u_1, \ldots, u_p$ in Krasinskiĭ's paper take on the roles of $\varepsilon_1, \ldots, \varepsilon_{\tilde{l}}$, and his conditions (3) and (9) guarantee the fulfillment of our conditions β and (3.25) respectively. According

to Krasinskiĭ's basic theorem, for every sufficiently small $\varepsilon_{\tilde{l}+1}$ there are values of the remaining parameters $\varepsilon_j = \mathring{\varepsilon}_j$ for which the system has an invariant k-dimensional torus with frequencies Ω. In fact in this case the family $\mathscr{F}^k_{n+1-\tilde{l}}$ lying in $\operatorname{Re}\mathscr{B}$ is $\mathscr{F}^k_2$, and for $a = 1$ it gives a one-parameter family of k-dimensional tori. According to Theorem 3.1 in the setting of Krasinskiĭ's basic theorem there exist conditionally periodic solutions with an arbitrary number of frequencies from k to $k + \tilde{l}$.

Remark. All the results of this section easily carry over to invertible systems in the non-resonant case. In that setting the normal form is

$$\dot{\check{U}} = \check{\Phi}(\check{P}, \underset{\sim}{E}) \ , \qquad \dot{\check{P}} = 0 \ .$$

where the variables are

$$\check{U} = (U, \check{u}_{k+1}, \ldots, \check{u}_{k+\tilde{l}}) \ , \qquad \check{P} \ ,$$

with $\check{u}_{k+j} = \arg(v_{k+j} + \mathrm{i}v_{k+\tilde{l}+j})$. Here the matrix $\tilde{B}_1$ is everywhere nilpotent (see the property 1 in the subsection 2.4.1); i.e. the matrix B is nilpotent outside of the coordinate subspaces. The set (3.19) is defined by the system $\check{\Phi}(\check{P}, \underset{\sim}{E}) = \Omega a$, which is analogous to system (3.22).

3.3. Resonances

We say that system (3.2) has a *resonance of multiplicity* $\tilde{k}$ if the equation $\mathrm{i}\langle P, \Omega\rangle + q_{l+1}\lambda_{l+1} + \cdots + q_{k+\tilde{l}}\lambda_{k+\tilde{l}} = 0$ has $\tilde{k}$ linearly independent solutions with integer $p_1, \ldots, p_k, q_{k+1}, \ldots, q_{k+\tilde{l}}$. Thanks to property (1.5), $\tilde{k} \leqslant \tilde{l}$. The non-resonant case ($\tilde{k} = 0$) was analyzed in the previous sections. We shall consider the case of a resonance of multiplicity 1, $\tilde{k} = 1$. We shall begin with the case $\tilde{l} = 1$.

$\underline{\tilde{l} = 1}$. The simplest case in this setting is when $k = 1$ and the initial torus $\mathscr{T}$ is a periodic solution. Resonance occurs if the number $\lambda = \lambda_{k+1}/(\mathrm{i}\omega_1)$ is rational: $\lambda = p/q$, where p and q are integers. The case $m = n = 0$ was examined in detail by Bruno [Bruno, 1970, 1972]. It turns out that for $|q| > 3$ three formal families of periodic solutions adjoin the initial torus $\mathscr{T}$: the basic family $\mathscr{F}^{(0)}$ corresponding to the coordinate subspace,

$$\mathscr{L} = \mathscr{L}[0] = \{U, V, \underset{\sim}{E}\colon v_{k+1} = \cdots = v_{k+2\tilde{l}} = 0\} \ ,$$

and two additional families, $\mathscr{F}^+$ consisting of unstable, and $\mathscr{F}^-$ consisting of stable periodic solutions. All of these families are of type $\mathscr{F}^1_1$, i.e., of one-parameter type (see figure 4). The solutions of the basic family have period close to the period $2\pi/\omega_1$ of the initial solution. The solutions of the other two families have periods near $2\pi q/\omega_1$. These two families form a formal set with an algebraic singularity at the initial solution $\mathscr{T}$. According to theorem 2.4 all of these families lie in the set $\mathscr{A}$, and according to theorem 2.2 these families are analytic.

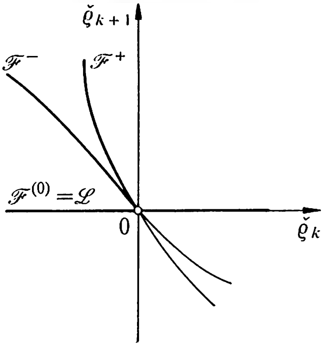

Fig. 4

All of these properties carry over to the general case of arbitrary m and n, except that now the families have $n + 1$ parameters. Finally, for any $k > 1$ the set $\operatorname{Re} \mathscr{A}$ consists of the same components $\mathscr{F}^{(0)} \subset \mathscr{L}[0]$, $\mathscr{F}^-$ and $\mathscr{F}^+$; they are all families of type $\mathscr{F}^k_{n+1}$. When the additional families $\mathscr{F}^+$ and $\mathscr{F}^-$ approach the basic family $\mathscr{F}^{(0)}$, tori of $\mathscr{F}^+$ and $\mathscr{F}^-$ tend to q-fold tori of $\mathscr{F}^{(0)}$. Now, however, the set $\mathscr{B} \subset \mathscr{A}$ is a set of analyticity. It may be shown that the set $\mathscr{B}$ lies in the basic component of the set $\mathscr{A}$,

$$\left\{U, V, \underset{\sim}{E}: \frac{\partial h}{\partial v_i} = \omega_i a, i = 1, \ldots, k; v_{k+1} = v_{k+2} = 0\right\},$$

and is defined in that component by the equation

$$\frac{\partial^2 h}{\partial v_{k+1} \partial v_{k+2}} = \lambda_{k+1} a .$$

That is, the set $\operatorname{Re} \mathscr{B}$ is the family $\mathscr{F}^k_n$. This family is a manifold if the determinant (3.31) does not vanish. In fact the family $\mathscr{F}^k_n$ is just that introduced by Moser [Moser, 1967, theorem 6]. Moreover, there are no other analytic families passing through the initial torus $\mathscr{T}$.

Now let us take up the case of "general position" with $\tilde{l} = 2$.

$\underline{\tilde{l} = 2}$. We shall limit our attention to the simplest case $k = m = n = 0$, i.e., to a Hamiltonian system with two degrees of freedom in the neighborhood of a stationary point. It is assumed that $\lambda = \lambda_1/\lambda_2$ is rational. Many studies have been made of the family of periodic solutions in this case [see, for example, Henrard, 1973, concluding remarks]. We shall briefly summarize the results. If λ and λ^{-1} are not integers then the normal form has one-parameter Lyapunov families (see example 4.2):

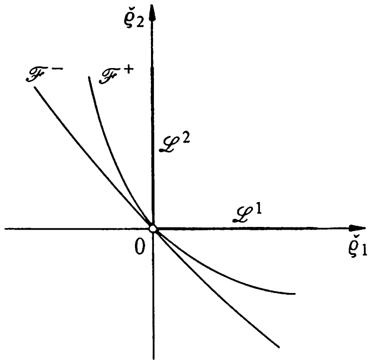

Fig. 5

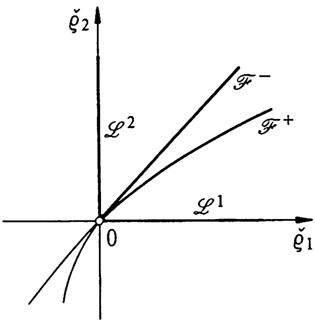

Fig. 6

$$\mathscr{L}^1 = \{V: v_2 = v_4 = 0\} \ , \qquad \mathscr{L}^2\{V: v_1 = v_3 = 0\} \ .$$

Moreover, there are two one-parameter families of periodic solutions $\mathscr{F}^-$ and $\mathscr{F}^+$. There are two cases (see (3.34)) depending on the signs of the coefficients of the normal form:

1) The families $\mathscr{F}^-$ and $\mathscr{F}^+$ are pure imaginary,
2) The families $\mathscr{F}^-$ and $\mathscr{F}^+$ have real parts.

In figures 5 and 6 the families $\mathscr{L}^1$, $\mathscr{L}^2$, $\mathscr{F}^-$, and $\mathscr{F}^+$ are represented in the $\check{\rho}_1$, $\check{\rho}_2$ plane for the respective cases 1) and 2) above. The heavy lines in these figures correspond to the real parts. If λ^{-1} or λ are integers different from 1, 2, then the families $\mathscr{L}^1$, $\mathscr{L}^2$, $\mathscr{F}^-$, and $\mathscr{F}^+$ remain; however, the family $\mathscr{L}^1$ (or $\mathscr{L}^2$), is not a manifold, but has an algebraic singularity at the origin together with the families $\mathscr{F}^-$ and $\mathscr{F}^+$. According to theorem 2.4

$$\operatorname{Re}\mathscr{A} = \operatorname{Re}\mathscr{L}^1 \cup \operatorname{Re}\mathscr{L}^2 \cup \operatorname{Re}\mathscr{F}^- \cup \operatorname{Re}\mathscr{F}^+ \ ,$$

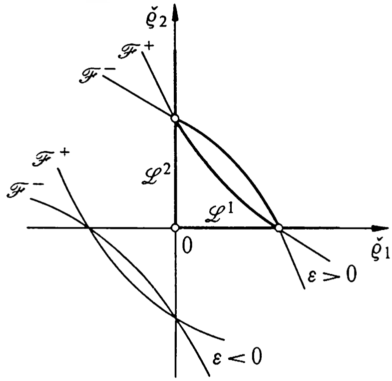

Fig. 7

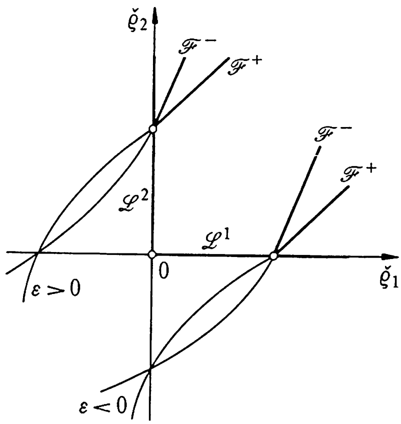

Fig. 8

and by theorem 2.2 all of these formal families are analytic since condition α is trivially satisfied [Meyer, et al, 1970, 1971; Henrard, 1973].

If there is a single small parameter ε in the system (i.e., if $n = 1$), then for each fixed small $\varepsilon \neq 0$ the families $\mathscr{L}^1$ and $\mathscr{L}^2$ remain where they were, but the families $\mathscr{F}^-$ and $\mathscr{F}^+$ change approximately as shown in figures 7 and 8 for the respective cases 1) and 2) (for $\varepsilon = 0$). Now in the case 1), new real branches of the families $\mathscr{F}^-$ and $\mathscr{F}^+$ appear. These are called "bridges" and connect the resonant periodic solutions of the families $\mathscr{L}^1$ and $\mathscr{L}^2$.

In the case of arbitrary $\tilde{k}$ we note that the number of parameters in the set $\mathscr{A}^*$ is $n + 1$, while the number in the set $\mathscr{B}$ is $n + 1 - \tilde{k}$.

§4. Families of Periodic Solutions

4.1. The Normal Form on an Integral Manifold

Let $0 \leqslant l' \leqslant l$ and suppose that in system (1.7),

$$\lambda_j \neq \mathrm{i}\langle P, \Omega\rangle + q_1\lambda_1 + \cdots + q_{l'}\lambda_{l'} , \qquad j = l' + 1, \ldots, l \tag{4.1}$$

for all $P \in \mathbb{Z}^k$ and non-negative integers $q_1, \ldots, q_{l'}$. According to property 2 of section 1.4 the normal form (1.12) has a formal integral manifold

$$\mathscr{L} = \mathscr{L}[\lambda_1, \ldots, \lambda_{l'}] = \{U, V, \underset{\sim}{E}: V'' = 0\} . \tag{4.2}$$

Here and below every vector $V = (v_1, \ldots, v_l)$ of dimension l will be divided into two subvectors $V' = (v_1, \ldots, v_{l'})$ and $V'' = \{v_{l'+1}, \ldots, v_l\}$ of dimensions l' and $l - l'$, respectively.

First assume that $m = 0$. On the manifold (4.2) the normal form (1.12) becomes the system

$$\begin{aligned} \dot{U} &= \Phi(U, V', 0, \underset{\sim}{E}) \equiv \hat{\Phi}(U, V', \underset{\sim}{E}) , \\ \dot{V}' &= \Psi'(U, V', 0, \underset{\sim}{E}) \equiv \hat{\Psi}'(U, V', \underset{\sim}{E}) , \end{aligned} \tag{4.3}$$

which is again a normal form. Let (1.11) be a normalizing transformation of (1.7) into (1.12). Instead of (1.11) let us consider the transformation

$$\begin{aligned} X &= U + \Xi(U, V', 0, \underset{\sim}{E}) \equiv U + \hat{\Xi} , \\ Y &= V + \underset{\sim}{H}(U, V', 0, \underset{\sim}{E}) \equiv V + \hat{\underset{\sim}{H}} . \end{aligned} \tag{4.4}$$

Here the variables U and V differ from the variables U and V of §1. Let the transformation (4.4) carry system (1.7) to the system

$$\begin{aligned} \dot{U} &= \Phi^*(U, V, \underset{\sim}{E}) , \\ \dot{V} &= \Psi^*(U, V, \underset{\sim}{E}) . \end{aligned} \tag{4.5}$$

Since transformation (4.4) and the normalizing transformation (1.11) coincide on

the manifold (4.2), then for $V'' = 0$ we have

$$\Phi^* = \hat{\Phi} \ , \qquad \Psi^{*\prime} = \hat{\Psi}' \ , \qquad \Psi^{*\prime\prime} = 0 \ .$$

Consequently, the manifold $V'' = 0$ is invariant for the system (4.5), and on that manifold system (4.5) induces the normal form (4.3). That is, in the expansions

$$\begin{aligned} \Phi^* &= \sum \Phi^*_{PQS} V^Q \underset{\sim}{E}^S \exp \mathrm{i}\langle P, U\rangle \ , \\ g_j^* &= \psi_j^*/v_j = \sum g^*_{jPQS} V^Q \underset{\sim}{E}^S \exp \mathrm{i}\langle P, U\rangle \ , \end{aligned} \tag{4.6}$$

the coefficients Φ^*_{PQS} and g^*_{jPQS} with $j \leqslant l'$ vanish if

$$Q'' = 0 \ , \qquad \mathrm{i}\langle P, \Omega\rangle + \langle Q', \Lambda'\rangle \neq 0 \ , \tag{4.7}$$

while $g^*_{jPQS} = 0$ for $j > l'$ if $q_j = -1$. A formal system (4.5) whose expansion (4.6) has property (4.7) is said to be *normalized on the manifold* (4.2). For example, the normal form (1.12) is normalized on all of its formal integral manifolds.

Theorem 4.1. *Suppose that in system* (1.7) $m = 0$ *and the vectors* Ω *and* Λ *satisfy inequalities* (4.1). *Then under restriction* 3 *there is a formal change of coordinates* (4.4) *which carries system* (1.17) *to system* (4.5), *normalized on the formal integral manifold* (4.2.).

If $m \neq 0$, then in § 1 system (1.7) was reduced to the seminormal form (1.17) by means of the transformation (1.16). We denote by $\hat{\Xi}$ and $\hat{\eta}_j$ the functions Ξ and η_j for $V'' = 0$ ($j = 1, \ldots, l + m$) and we carry out the transformation

$$\begin{aligned} X &= U + \hat{\Xi}(U, V', W, \underset{\sim}{E}) \ , \\ y_i &= v_i + \hat{\eta}_i(U, V', W, \underset{\sim}{E}) \ , \qquad i = 1, \ldots, l \ , \\ y_{l+j} &= w_j + \hat{\eta}_{l+j}(U, V', W, \underset{\sim}{E}) \ , \qquad j = 1, \ldots, m \ . \end{aligned} \tag{4.8}$$

The coordinates U, V, W differ from the analogous coordinates of § 1. As a result of the transformation (4.8), system (1.7) is carried into the system

$$\begin{aligned} \dot{U} &= \Phi^*(U, V, \underset{\sim}{E}) + \check{\Phi}^*(U, V, W, \underset{\sim}{E}) \ , \\ \dot{V} &= \Psi^*(U, V, \underset{\sim}{E}) + \check{\Psi}^*(U, V, W, \underset{\sim}{E}) \ , \\ \dot{W}_- &= \underset{\sim}{X}^*_-(U, V, W_-, \underset{\sim}{E}) + \check{\underset{\sim}{X}}^*_-(U, V, W, \underset{\sim}{E}) \ , \\ \dot{W}_+ &= \underset{\sim}{X}^*_+(U, V, W_+, \underset{\sim}{E}) + \check{\underset{\sim}{X}}^*_+(U, V, W, \underset{\sim}{E}) \ , \end{aligned} \tag{4.9}$$

where the right hand sides are in the ring $\mathscr{P}_U[[V, W, \underset{\sim}{E}]]$. For $V'' = 0$ all the starred functions in system (4.9) coincide with the corresponding unstarred functions in the seminormal form (1.17). We say that the formal system (4.9) is

seminormalized on the manifold (4.2) if

1) The series expansions (4.6) for Φ^* and Ψ^* have property (4.7);

2) For $V'' = 0$ the series $\hat{\Phi}^*$, $\hat{\Psi}^*$, $\tilde{X}^*$, and $\underset{\sim}{X}^*$ have properties 2 and 3 in the definition of a seminormal form (adding asterisks to the terms in that definition as appropriate).

Therefore, for $V'' = 0$ system (4.9), seminormalized on manifold (4.2), also has the same properties as the seminormal form (1.17). We note three of them:

1. The manifold $\mathscr{L}_W = \{U, V, W, \underset{\sim}{E}\colon V'' = 0\}$ is not, generally speaking, an integral manifold.

2. The manifolds

$$\mathscr{L}_- = \{U, V, W, \underset{\sim}{E}\colon V'' = 0, W_+ = 0\} \ ,$$

$$\mathscr{L}_+ = \{U, V, W, \underset{\sim}{E}\colon V'' = 0, W_- = 0\}$$

are integral manifolds. On each manifold system (4.9) induces a "triangular" system of type (1.21) with variables V' instead of V. The manifolds $\mathscr{L}_j = \mathscr{L}_W \cap \mathscr{W}_j$ are also invariant.

3. $\mathscr{L} = \mathscr{L}_- \cap \mathscr{L}_+$ is an integral manifold on which system (4.9) induces the normal form (4.3).

Theorem 4.2. *If, in system* (1.7), Ω *and* Λ' *satisfy inequalities* (4.1), *then under restriction* 3 *there is a formal change of coordinates* (4.8) *which reduces system* (1.7) *to system* (4.9), *seminormalized on the formal integral manifold* (4.2.).

4.2. Questions of Convergence

Let us first consider the classical formulation of the question of the convergence of transformation (4.4) or (4.8):

Problem 4.1. *Under what conditions on the normal form* (4.3) (*or, more generally, on* (1.12)) *is the manifold* $\mathscr{L}$ (*or* $\mathscr{L}_-$ *or* $\mathscr{L}_+$) *analytic?*

Condition α′. *The numbers* $i\omega_1, \ldots, i\omega_k, \lambda_1, \ldots, \lambda_{l'}$ *are pairwise commensurable.* 0 *is considered to be commensurable with every number, including zero.*

This condition guarantees that there are no small divisors in transformation (4.8).

Condition A′. *There is a series* $a = a(U, V', \underset{\sim}{E})$ *such that in the normal form* (4.3)

$$\hat{\Phi} = \Omega a \ , \qquad \hat{\psi}_j = \lambda_j v_j a \ , \qquad j = 1, \ldots, l' \ .$$

Theorem 4.3. *In the setting of theorem* 4.2 *suppose that* Ω *and* Λ' *satisfy condition* α' *and the normal form* (4.3) *satisfies condition* A'. *Then transformation*

(4.8) *converges in some neighborhood of the initial torus* $\mathcal{T}$. *Hence, the manifolds* $\mathcal{L}$, $\mathcal{L}_-$, *and* $\mathcal{L}_+$ *are analytic.*

The advantage of this theorem over theorem 2.1 is in the weakened conditions on the eigenvalues and on the normal form. On the other hand, analytic transformations are obtained only in certain subspaces. Note that thanks to condition (1.5) we have that $k = 0$ or 1 and restriction 3 is automatically satisfied. For $k = m = 0$ theorem 4.3 is a consequence of results of Bruno [Bruno, 1971, theorem 3 in § 10; or 1974a, theorem 1, which contains all earlier cases]. That condition A′ is necessary for the analyticity of the manifold $\mathcal{L}$ was shown by Bruno [Bruno, 1974a, theorem 2; also 1971, theorem III].

Example 4.1. Let $k = 1$ and suppose that $\lambda_1, \ldots, \lambda_l$ are all different from 0. Then in the subspace

$$\mathcal{L} = \mathcal{L}[0] = \{U, V, W, \underset{\sim}{E}\colon V = 0, W = 0\}$$

condition A′ is

$$\hat{\varphi}_1(\underset{\sim}{E}) = a\omega_1 \ ,$$

and is always satisfied. According to theorem 4.3 the periodic solution $\mathcal{T}$ may be continued analytically in the parameters $\underset{\sim}{E}$ if all of its eigenvalues are different from zero. This is Poincaré's theorem [Poincaré, 1892, v. 1, Ch. 3].

Example 4.2. Suppose that the original system is the Hamiltonian system (3.1), $k = 0$, the pure imaginary numbers $\lambda_1 = -\lambda_2 \neq 0$, and the numbers $\lambda_3/\lambda_1, \ldots, \lambda_l/\lambda_1$ are not integers. Then the subspace $\mathcal{L} = \mathcal{L}[\lambda_1, \lambda_2]$ is invariant in the normal form (3.6). The normal form is also Hamiltonian on $\mathcal{L}$:

$$\dot{v}_1 = \frac{\partial \hat{h}}{\partial v_2}\,, \qquad \dot{v}_2 = -\frac{\partial \hat{h}}{\partial v_1}\,, \tag{4.10}$$

where

$$\hat{h} = h|_{v_3 = \cdots = v_l = 0} = \sum h_{q_1 q_2} v_1^{q_1} v_2^{q_2} \ .$$

Here the summation is over those integers $q_1, q_2 \geqslant 0$ which satisfy the equation $q_1\lambda_1 + q_2\lambda_2 = 0$. Since $\lambda_1 = -\lambda_2$ we have that $q_1 = q_2$. Thus, $\hat{h}$ is a series in non-negative integral powers of $\rho = v_1 v_2$. The normal form (4.10) has the form

$$\dot{v}_1 = v_1 \frac{\partial \hat{h}}{\partial \rho}\,, \qquad v_2 = -v_2 \frac{\partial \hat{h}}{\partial \rho} \ .$$

Condition A′ is satisfied with $a = \dfrac{\partial \hat{h}}{\lambda_1 \partial \rho}$. According to theorem 4.3 the manifold $\mathcal{L}$ is analytic and contains an $(n + 1)$-parameter family of periodic solutions adjoining the initial stationary point $\mathcal{T}$. This is the theorem of Lyapunov [Lyapunov, 1892, Ch. II, 33–41]. See also some remarks of Bruno [Bruno, 1971, § 10, example 3].

Problem 4.2. *On which formal set $\mathscr{M}'$ does the transformation* (4.8) *converge?*

Let us define the set

$$\mathscr{A}'_W = \{U, V, W, \underset{\sim}{E}\colon V'' = 0, \hat{\Phi} = \Omega a, \hat{\psi}_j = \lambda_j v_j a, j = 1, \ldots, l'\}$$

where a is a free parameter. Evidently we have that $\mathscr{A}'_W = \mathscr{L}_W \cap \mathscr{A}_W$.

Theorem 4.4. *Suppose that Ω and Λ' satisfy condition α' in the setting of theorem 4.2. Then the transformation* (4.8) *converges near the initial torus $\mathscr{T}$ in the set $\mathscr{A}'_W$, which is itself an analytic set.*

Let us suppose that condition α' is satisfied for the manifold $\mathscr{L}$ and that theorem 4.4 is applicable. If condition α is satisfied, then theorem 4.4 is a particular case of theorem 2.2. If condition α is not satisfied, but condition β is satisfied, then theorem 2.3 gives on $\mathscr{L}_W$ only the analyticity of the set $\mathscr{B}_W \cap \mathscr{L}_W$, which is smaller than the set $\mathscr{A}'_W$. Finally, it is possible that condition β is not satisfied and condition α' is. In the last two cases theorem 4.4 gives new results in comparison with theorems 2.2 and 2.3.

4.3. Periodic Solutions

In subsection 2.3.2 there was introduced for each coordinate subspace $\mathscr{K}$ or $\mathscr{L}$ an index of irrationality $r' = r'(\mathscr{K})$, the maximal number of elements of the set $\{\mathbf{i}\omega_1, \ldots, \mathbf{i}\omega_k; \lambda_1, \ldots, \lambda_{l'}\}$ which are linearly independent over the rational field. Thanks to (1.5), we have that $r' \geqslant k$. Condition α' is satisfied only for those submanifolds $\mathscr{L}$ for which the index of irrationality is 0 or 1.

First let us consider the case $r' = 0$, which occurs only for $k = 0$ when the torus $\mathscr{T}$ is a stationary point. Let the manifold $\mathscr{L} = \mathscr{L}[0]$; that is,

$$\lambda_1 = \cdots = \lambda_{l'} = 0\ , \qquad \lambda_j \neq 0\ , \qquad j > l'\ .$$

On the manifold $\mathscr{L}_W$ the set $\mathscr{A}'_W$ is defined by the system of equations $\hat{\Psi}' = 0$, i.e., the set $\mathscr{A}' = \mathscr{A}'_W \cap \mathscr{L}$ is a formal family of stationary points. According to theorem 4.4, it is analytic.

Now let us consider the manifold $\mathscr{L}$ for which $r'(\mathscr{L}) = 1$. According to theorem 2.4, the set $\operatorname{Re} \mathscr{A}'$ consists of a family of periodic solutions adjoining the initial solution $\mathscr{T}$. By theorem 4.4 the set $\mathscr{A}'$ is analytic, and we have the following consequence:

Corollary 4.1. *All formal families of periodic solutions adjoining the initial torus $\mathscr{T}$ and lying outside the manifold $\mathscr{L}[0]$ for $k = 0$ are analytic.*

It was shown in example 2.3 that for $k = 0$ there may be a formal non-analytic family of periodic solutions in the subspace $\mathscr{L}[0]$. In an earlier paper [Bruno,

1970] it was hypothesized that all formal families of periodic solutions adjoining $\mathcal{T}$ are analytic. Now it is clear that this hypothesis is true only if the formal families in $\mathcal{L}[0]$ for $k = 0$ are excluded. In another paper [Bruno, 1974a, theorem 3] the analyticity of certain families is asserted. These are the formal families of periodic solutions which are manifolds and for which the period can be expanded in a Taylor series in the variables V, $\underset{\sim}{E}$. This property is only possessed by families lying outside $\mathcal{L}[0]$ for $k = 0$. But for families lying in $\mathcal{L}[0]$ with $k = 0$, the period has a singularity of pole type at the stationary point; i.e., in these families the period tends to infinity near the point $\mathcal{T}$. For example, the period of the solution (2.13) in example 2.3 is

$$\frac{2\pi}{4\sqrt{h}} = \frac{\pi}{2(v_1^2 + v_2^2)} .$$

Problem 4.3. *Find all families of periodic solutions adjoining* $\mathcal{T}$.

In order to solve this problem it is at first necessary to find all formal integral subspaces $\mathcal{L}$ with index of irrationality $r'(\mathcal{L}) = 1$. Then in each of these subspaces it is necessary to find the set $\operatorname{Re}\mathcal{A}'$. If $k = 0$ and some $\lambda_j = 0$, then there is some ambiguity about the formal families of periodic solutions $\mathcal{L}[0]$. In all other cases the indicated solution of problem 4.3 is complete.

It is also possible to consider the complex version of problem 4.3. Let $Z = \underset{\sim}{Z}(Z^0, \Delta^0, t)$ be the general complex solution of system (1.1) which has value Z^0 for $t = 0$ and $\Delta = \Delta^0$. This solution is periodic (or contains a "complex cycle"), if for some complex number $t^0 \neq 0$ the equality $\underset{\sim}{Z}(Z^0, \Delta^0, t^0) = Z^0$ is satisfied. The number t^0 is the period of this solution. It is easy to see that for every subspace $\mathcal{K}$ with index of irrationality $r'(\mathcal{K}) = 1$ the set $\mathcal{A} \cap \mathcal{K}$ consists of complex periodic solutions (cf. theorem 2.4).

Example 4.3. Let $k = 0$, $l \geqslant 2$, $n = 1$, m arbitrary; $m_1\lambda_1 + m_2\lambda_2 = 0$, where m_1 and m_2 are relatively prime positive integers and $\lambda_j \neq q_1\lambda_1 + q_2\lambda_2, j = 3, \ldots, l$, for all integers $q_1, q_2 \geqslant 0$. In the integral subspace

$$\mathcal{L} = \mathcal{L}[\lambda_1, \lambda_2] = \{V, \varepsilon \colon v_3 = \cdots = v_l = 0\}$$

the normal form (4.3) has the form (2.15′)

$$\dot{v}_j = v_j \sum_{k=0}^{\infty} g_{jk}(\varepsilon)(v_1^{m_1} v_2^{m_2})^k \equiv v_j g_j , \qquad j = 1, 2 .$$

It was shown in example 2.4 that the set $\mathcal{A}$ for this system consists of three components $\mathcal{A}^1$, $\mathcal{A}^2$, and $\mathcal{A}^3$. If condition (2.16) holds, the set $\mathcal{A}^3$ is a manifold of the form $\varepsilon = \mathring{\varepsilon}(v_1^{m_1} v_2^{m_2})$. The quantity $\tilde{v} = v_1^{m_1} v_2^{m_2}$ is constant on each solution in the set $\mathcal{A}^3$. According to (2.6) the solutions in $\mathcal{A}^3$ are complex periodic "cycles" of period

$$\frac{2\pi i m_1}{g_2(\bar{v}, \mathring{\varepsilon})}\,.$$

By theorem 4.4 the component $\mathscr{A}^3$ is an analytic family of cycles. If a new time $\tilde{t} = \tau t$ is introduced as in example 2.4, where τ is some complex number, then we obtain new eigenvalues $\tilde{\lambda}_j = \lambda_j/\tau$. On the complex plane the points $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ are found on some line passing through the origin. The analyticity of the set $\mathscr{A}^3$ is preserved and we obtain theorem 1 of Pyartli [Pyartli, 1972]. If the original system is real and the eigenvalues λ_1 and λ_2 are pure imaginary, then the periodic solutions of the family $\mathscr{A}^3$ are real.

§ 5. Integral Manifolds with Small Divisors

Now let us consider the formal integral manifold (4.2) for which the condition α' is not satisfied and $l' < l$. Thus, in the calculation of this manifold small divisors appear, and the normal form (4.3) on the manifold (4.2) is no longer sufficient to answer the questions raised in problems 4.1 and 4.2.

5.1. Formal Analysis

First let $m = 0$. In the normal form (1.12) we isolate the terms of Ψ'' which are linear in V'':

$$\psi_i = \sum_{j=l'+1}^{l} c_{ij} v_j + \cdots \ , \qquad i = l' + 1, \ldots, l \ ,$$

where c_{ij} depend only on U, V, $\underset{\sim}{E}$, and we form the square matrix $C'' = (c_{ij})_{l'+1}^{l}$, where $i, j = l' + 1, \ldots, l$. The system

$$\begin{aligned} \dot{U} &= \hat{\Phi}(U, V', \underset{\sim}{E}) \ , \\ \dot{V}' &= \hat{\Psi}'(U, V', \underset{\sim}{E}) \ , \\ \dot{V}'' &= C''(U, V', \underset{\sim}{E}) V'' \end{aligned} \tag{5.1}$$

is called the *complete normal form of the integral manifold* (4.2) [Bruno, 1971, § 10]. The right hand sides of system (5.1) are made up of those terms from the right hand sides of system (1.12) for which $\|Q'\| \leqslant 0$ (in the notation of (1.13) and (1.14)), i.e., either $Q'' = -E_i$ or $Q'' = E_i - E_j$, $i, j = l' + 1, \ldots, l$. Now we shall apply to system (1.7) the transformation

$$\begin{aligned} X &= U + \hat{\Xi}(U, V', \underset{\sim}{E}) \ , \\ Y' &= V' + \underset{\sim}{\hat{H}}'(U, V', \underset{\sim}{E}) \ , \\ Y'' &= V'' + \underset{\sim}{\hat{H}}''{}^{*}(U, V', \underset{\sim}{E}) \ . \end{aligned} \tag{5.2}$$

The coordinates U, V here are different from the coordinates U, V in § 1, but

the transformation (5.2) is connected with the normalizing transformation (1.11) by the equalities

$$\hat{\Xi} = \Xi \ , \qquad \hat{\underset{\sim}{H}}' = \underset{\sim}{H}' \quad \text{for} \quad V'' = 0 \ ;$$

i.e., $\hat{\Xi}$ and $\hat{\underset{\sim}{H}}'$ are the terms of Ξ and $\underset{\sim}{H}'$ which are independent of V'', while $\hat{\underset{\sim}{H}}''^{*}$ is the part of the series $\underset{\sim}{H}''$ which is constant or linear in V''. Such a transformation reduces system (1.7) to the form

$$\begin{aligned} \dot{U} &= \Phi^{**}(U, V, \underset{\sim}{E}) \ , \\ \dot{V} &= \Psi^{**}(U, V, \underset{\sim}{E}) \ , \end{aligned} \tag{5.3}$$

where for $V'' = 0$ there is obtained the complete normal form (5.1) on the manifold $\mathscr{L}$:

$$\Phi^{**} = \hat{\Phi} \ , \qquad \Psi^{**\prime} = \hat{\Psi}' \ , \qquad \frac{\partial \Psi^{**\prime\prime}}{\partial V''} = C'' \ . \tag{5.4}$$

If we set

$$\begin{aligned} \Phi^{**} &= \sum \Phi^{**}_{PQS} V^Q \underset{\sim}{E}^S \exp \mathrm{i}\langle P, U \rangle \ , \\ g_j^{**} = \psi_j^{**}/v_j &= \sum g_{jPQS} V^Q \underset{\sim}{E}^S \exp \mathrm{i}\langle P, U \rangle \ , \qquad j = 1, \ldots, l \ , \end{aligned} \tag{5.5}$$

then

$$\begin{aligned} \Phi^{**}_{PQS} = 0 \quad \text{and} \quad g^{**}_{jPQS} = 0 \ , \qquad \text{if} \\ \|Q''\| \leqslant 0 \ , \qquad \mathrm{i}\langle P, \Omega \rangle + \langle Q, \Lambda \rangle \neq 0 \ . \end{aligned} \tag{5.6}$$

That is, among the coefficients Φ^{**}_{PQS} and g^{**}_{jPQS} with $\|Q''\| \leqslant 0$ only the resonant terms for which equality (1.15) is satisfied are different from zero. A system (5.3) with this property is said to *completely normalized on the manifold* (4.2).

Restriction 3′: *Let* $\gamma(Q) \geqslant 0$ *for all vectors* $Q \in \mathbb{N}^l$, *with* $\|Q''\| \leqslant 0$ (*the quantity* $\gamma(Q)$ *was introduced in section* 1.4 *before restriction* 3).

Evidently, restriction 3′ is weaker than restriction 3; for $k = 0$ or 1 it is automatically satisfied.

Theorem 5.1. *Suppose that the vectors* Ω *and* Λ *in system* (1.7) *with* $m = 0$ *satisfy inequality* (4.1). *Then under restriction* 3′ *there is a formal change of coordinates* (5.2) *reducing system* (1.7) *to system* (5.3), *which is completely normalized on the formal integral manifold* (4.2).

If $m \neq 0$, then in § 1 system (1.7) was reduced to the seminormal form (1.17) by means of transformation (1.16). Let us denote by $\hat{\Xi}$ and $\hat{\eta}_j$ the series Ξ and η_j for $V'' = 0$, respectively, while $\hat{\eta}_j^*$ denotes the terms in the series η_j which are constant or linear in V''. Now together with the transformation (1.16) we make the transformation

$$
\begin{aligned}
X &= U + \hat{\Xi}(U, V', W, \underset{\sim}{E}) \ , \\
y_i &= v_i + \hat{\eta}_i(U, V', W, \underset{\sim}{E}) \ , \qquad i = 1, \dots, l' \ , \\
y_j &= v_j + \hat{\eta}_j^*(U, V', V'', W, \underset{\sim}{E}) \ , \qquad j = l' + 1, \dots, l \ , \\
y_{l+k} &= w_k + \hat{\eta}_{l+k}(U, V', W, \underset{\sim}{E}) \ , \qquad k = 1, \dots, m \ .
\end{aligned}
\tag{5.7}
$$

This transformation reduces (1.7) to the form

$$
\begin{aligned}
\dot{U} &= \Phi^{**}(U, V, \underset{\sim}{E}) + \check{\Phi}^{**}(U, V, W, \underset{\sim}{E}) \ , \\
\dot{V} &= \Psi^{**}(U, V, \underset{\sim}{E}) + \check{\Psi}^{**}(U, V, W, \underset{\sim}{E}) \ , \\
\dot{W} &= \underset{\sim}{X}^{**}(U, V, W, \underset{\sim}{E}) + \check{\underset{\sim}{X}}^{**}(U, V, W, \underset{\sim}{E}) \ ,
\end{aligned}
\tag{5.8}
$$

where the right hand terms lie in the ring $\mathscr{P}_U[[V, W, \underset{\sim}{E}]]$. A system of form (5.8) is said to be *completely seminormalized on the manifold* (4.2) if

1) it is seminormalized on the manifold (4.2);

2) the series Φ^{**} and Ψ^{**} have properties (5.4) and (5.6). System (5.8), since it is completely seminormalized on the manifold (4.2), has all the properties of the simple seminormalized system (4.9).

Theorem 5.2. *Suppose that Ω and Λ in system* (1.7) *satisfy inequality* (4.1). *Then under restriction 3′ there is a formal change of coordinates* (5.7) *reducing system* (1.7) *to system* (5.8), *which is completely seminormalized on the formal integral manifold* (4.2).

Note that theorems 5.1 and 5.2 are true under restriction 3′ which is weaker than restriction 3 used in theorems 1.1 and 1.2. Therefore there is the situation when system (1.7) does not reduce to the seminormal form (1.17) since restriction 3 does not hold, but it does reduce to system (5.8), which is completely seminormalized on the manifold (4.2) if restriction 3′ is satisfied.

5.2. Questions of Convergence

First let us answer the question posed in problem 4.1. Let $\alpha'_j = \min |\mathbf{i}\langle P, \Omega\rangle + \langle Q, \Lambda\rangle|$, where $P \in \mathbb{Z}^k$, $Q \in \mathbb{N}^l$, $\mathbf{i}\langle P, \Omega\rangle + \langle Q, \Lambda\rangle \neq 0$, $|P| + \|Q\| < 2^j$, $j = 1, 2, \dots$, $\|Q''\| \leqslant 0$. Let $\beta' = \sum_1^\infty 2^{-j} \ln \alpha'_j$.

Condition β': $\beta' > -\infty$.

The condition $\|Q''\| \leqslant 0$ distinguishes the number α'_j from the number α_j. That is, in calculating α'_j the minimum is taken only over those Q for which Q'' is 0, $-E_i$, or $E_i - E_j$ $(i, j \geqslant 1)$. Therefore, $\alpha_j \leqslant \alpha'_j$ and it may be that condition β' holds if condition β does not. Note that condition β' is a condition on all

numbers $\omega_1, \ldots, \omega_k, \lambda_1, \ldots, \lambda_l$. If condition β' is satisfied, then restriction 3′ is also satisfied.

Condition B. *In the complete normal form* (5.1) *condition A' is satisfied, and the matrix*

$$B'' = C'' - L''a \tag{5.9}$$

is nilpotent, where $L'' = \{\lambda_{l'+1}, \ldots, \lambda_l\}$ is a diagonal matrix.

Theorem 5.3. *Suppose that in the setting of theorem* 5.2 *condition β' is satisfied and the complete normal form* (5.1) *satisfies condition B. Then the transformation* (5.7) *is analytic in some neighborhood of the initial torus $\mathscr{T}$.*

This theorem is similar to a theorem of Bruno [Bruno, 1971, theorem 4 in § 10] which was formulated for the situation covered by theorem 5.1 for $k = 0$. The necessity of condition A' and the nilpotency of the matrix B'' was shown in the reference just cited and also in other papers [Bruno, 1974a, theorem 2; 1974b].

Now let us turn to problem 4.2. Consider the set $\mathscr{A}'$ on the manifold (4.2) the matrix

$$B' = \begin{pmatrix} \dfrac{\partial \hat{\Phi}}{\partial U} & \dfrac{\partial \hat{\Phi}}{\partial V'} \\[2ex] \dfrac{\partial \hat{\Psi}'}{\partial U} & \dfrac{\partial \hat{\Psi}'}{\partial V'} - L'\alpha \end{pmatrix}$$

where $L' = \{\lambda_1, \ldots, \lambda_{l'}\}$ is a diagonal matrix; B' and the matrix B'' in (5.9) are defined on $\mathscr{A}'$. The set $\mathscr{B}'$ is that subset of the set $\mathscr{A}'$ on which the matrix B' is nilpotent. The set $\mathscr{B}''$ is that subset of $\mathscr{B}'$ on which the matrix (5.9) is nilpotent. It is clear that

$$\mathscr{L} \supset \mathscr{A}' \supset \mathscr{B}' \supset \mathscr{B}'' .$$

If restriction 3 is satisfied, then we have the seminormal form (1.17) for which the set $\mathscr{B}$ is defined. According to subsection 2.4.2, $\mathscr{B}'' = \mathscr{B} \cap \mathscr{L}$. We set $\mathscr{B}''_W = \{U, V, W, \underset{\sim}{E}\colon U, V, \underset{\sim}{E} \in \mathscr{B}''\}$.

Theorem 5.4. *Suppose that in the setting of theorem* 5.2 *condition β' is satisfied. Then transformation* (5.7) *converges on the set $\mathscr{B}''_W$ near the initial torus $\mathscr{T}$, and $\mathscr{B}''_W$ is an analytic set.*

Since $\mathscr{B}_W \supset \mathscr{B}''_W$ then, under condition β, theorem 5.4 is a consequence of theorem 2.3. The advantage of theorem 5.4 lies solely in that it uses the weaker condition β' on Ω and Λ (instead of condition β of theorem 2.3). Note that theorem 5.3 is a particular case of theorem 5.4. In fact, if condition A' is satisfied then according to example 2.1 the matrix B' is nilpotent on the manifold $\mathscr{L}$;

therefore $\mathscr{A}' = \mathscr{B}' = \mathscr{L}$. Thanks to condition B we have that $\mathscr{L} = \mathscr{B}''$ and theorem 5.4 is applicable.

5.3. Continuation of the Initial Torus

Let l' be such that

$$\begin{aligned} &\lambda_i = \mathbf{i}\langle P^{(i)}, \Omega\rangle \ , \qquad && P^{(i)} \in \mathbb{Z}^k \ , \qquad && i \leqslant l' \ , \\ &\lambda_j \neq \mathbf{i}\langle P, \Omega\rangle \ , && \text{all } P \in \mathbb{Z}^k \ , && j > l' \ . \end{aligned} \tag{5.10}$$

For simplicity we assume that $\lambda_1 = \cdots = \lambda_{l'} = 0$; this assumption does not restrict the generality of our results [Moser, 1967, introduction]. The normal form (1.12) has an integral manifold (4.2), which we denote by $\mathscr{L}[0]$ and call the *null manifold*. Now consider the set $\mathscr{B} \cap \mathscr{L}[0]$. The complete normal form (5.1) on the null manifold has the form

$$\begin{aligned} \dot{U} &= \hat{\Phi}(V', \underset{\sim}{E}) \ , \\ \dot{V}' &= \hat{\Psi}'(V', \underset{\sim}{E}) \ , \\ \dot{V}'' &= C''(U, V', \underset{\sim}{E}) V'' \ . \end{aligned} \tag{5.11}$$

Let us make the additional assumption

$$\lambda_i - \lambda_j \neq \mathbf{i}\langle P, \Omega\rangle \ , \qquad \text{for all} \quad P \in \mathbb{Z}^k \ , \qquad P \neq 0 \ , \qquad i, j = l' + 1, \ldots, l \ . \tag{5.12}$$

Then the matrix C'' does not depend on U and commutes with the diagonal matrix $L'' = \{\lambda_{l'+1}, \ldots, \lambda_l\}$:

$$C'' = C''(V', \underset{\sim}{E}) \ , \tag{5.13}$$

$$C''L'' = L''C'' \ . \tag{5.14}$$

This follows from the fact that the only non-zero coefficients $c_{ijPQ'S}$ in the series expansion of an element c_{ij} of C'',

$$c_{ij} = \sum c_{ijPQ'S} V'^{Q'} \underset{\sim}{E}^S \exp \mathbf{i}\langle P, U\rangle$$

are those for which

$$\lambda_j - \lambda_i = \mathbf{i}\langle P, \Omega\rangle + \langle Q', \Lambda'\rangle \ . \tag{5.15}$$

As long as $\Lambda' = 0$ and (5.12) is satisfied then equalities (5.15) hold only for $P = 0$. Hence, (5.13) is satisfied and $c_{ij} = 0$ if $\lambda_i \neq \lambda_j$, $i, j = l' + 1, \ldots, l$; i.e., (5.14) is satisfied.

The set $\mathscr{A}'$ is defined by the system

$$\hat{\Phi}(V', \underset{\sim}{E}) = \Omega a \ , \tag{5.16}$$

$$\hat{\Psi}'(V', \underset{\sim}{E}) = 0 \ . \tag{5.17}$$

According to theorem 2.4 the set Re $\mathscr{A}'$ consists of families of k-dimensional tori adjoining the torus $\mathscr{T}$. The set $\mathscr{B}''$ is defined by conditions (5.16), (5.17) and the requirement that the matrices

$$B_1' = \frac{\partial \hat{\Psi}'}{\partial V'}, \qquad B'' = C'' - L''a \tag{5.18}$$

be nilpotent. The number of variables V', $\underset{\sim}{E}$, a is $l' + n + 1$, while the number of conditions is $k + l' + l' + (l - l') = k + l' + l$. Hence, the number of free parameters in the set $\mathscr{B}''$ is $l' + n + 1 - (k + l' + l) = n + 1 - (k + l)$. That is, in an arbitrary system sufficiently many of the original parameters are required so that the set $\mathscr{B}''$ may be distinguished from the initial torus $\mathscr{T}$.

As seen from section 3.2, $l' = k$ and $\hat{\Psi}' = 0$ for a Hamiltonian system in general position. Therefore the set $\mathscr{B}''$ is defined only by system (5.16) and by the nilpotency of the matrix B''. The matrix B'' of order $2\tilde{l} = l - k$ is Hamiltonian, and its characteristic polynomial contains only even powers (i.e., only $\tilde{l}$ nontrivial coefficients). Hence, the nilpotency of the matrix B'' imposes only $\tilde{l}$ conditions. Together with system (5.16) the set $\mathscr{B}''$ is defined by $k + \tilde{l}$ equations in $k + n + 1$ parameters. That is the number of free parameters is

$$k + n + 1 - (k + \tilde{l}) = n + 1 - \tilde{l} .$$

If $\tilde{l} = 0$, then the set $\mathscr{B}''$ always has at least one parameter (see examples 3.1 and 3.2). If $\tilde{l} = 1$ and $n = 1$, then the set $\mathscr{B}''$ has a single parameter (see section 3.2 with $\tilde{l} = 1$ and $n = 1$). In particular if the determinant of the matrix (3.28) (or of (3.31), or (3.32)) is different from zero, then the set Re $\mathscr{B}''$ also has one parameter. According to theorem 5.4 the initial torus $\mathscr{T}$ may be continued analytically in the parameter ε if conditions (5.10), (5.12), β', and (3.28) hold. Moser has considered the particular case of this assertion for $m = 0$ [Moser, 1967, theorem 6]. Inequalities (5.10) and (5.12) hold under condition (1.7), condition (β') holds under condition (1.7′), while condition (3.28) is Moser's condition (6.16). In comparison with the situation in section 3.B there are now weaker conditions on Ω and Λ: conditions (3.15) and β are replaced by (5.10) and (5.12), and β', respectively.

Moser suggested the analytic continuation of the initial torus $\mathscr{T}$ with respect to the small parameter ε, introducing into the system a certain number of new parameters for this purpose [Moser, 1967; also, 1966, 1969]. To do this he looked for those continuations with all frequencies and eigenvalues exactly the same as those of the initial torus $\mathscr{T}$. More recently it has been observed [Bibikov, 1973a] that it is not necessary to preserve the complex eigenvalues, only to preserve the frequencies and the eigenvalues lying on the imaginary axis. This makes it possible to decrease the number of additional parameters introduced into the system.

Now let us consider the Moser-Bibikov construction from our point of view. We shall write the original system (1.7) in the form

$$\begin{aligned} \dot{X} &= \Omega + \varepsilon F^{(1)}(X, Y, Z, \varepsilon) , \\ \dot{Y} &= LY + \varepsilon F^{(2)}(X, Y, Z, \varepsilon) , \\ \dot{Z} &= A_1 Z + \varepsilon F^{(3)}(X, Y, Z, \varepsilon) , \end{aligned} \tag{5.19}$$

where diagonal matrix $L = \{\lambda_1, \ldots, \lambda_l\}$, $\operatorname{Re}\lambda_j = 0$, the eigenvalues of the matrix A_1 are $\mu_1, \ldots, \mu_m$, $\operatorname{Re}\mu_j \neq 0$. Instead of system (5.19) we shall consider system

$$\begin{aligned} \dot{X} &= \Omega + \underset{\sim}{K} + \varepsilon F^{(1)} , \\ \dot{Y} &= LY + \underset{\sim}{N} + MY + \varepsilon F^{(2)} , \\ \dot{Z} &= A_1 Z + \varepsilon F^{(3)} , \end{aligned} \tag{5.20}$$

where $\underset{\sim}{K} = (\kappa_1, \ldots, \kappa_k)$, $\underset{\sim}{N} = (v_1, \ldots, v_l)$ are vectors and M is a square matrix of order l, where

$$L\underset{\sim}{N} = 0 , \qquad LM = ML \tag{5.21}$$

On the manifold $\mathscr{L}[0]$ system (5.20) has the complete normal form (5.1) whose right hand sides are

$$\begin{aligned} \hat{\Phi} &= \Omega + \underset{\sim}{K} + O(\varepsilon) + O(|V'|) + O(|\underset{\sim}{E}|^2) \\ \hat{\Psi}' &= \underset{\sim}{N}' + M'V' + O(\varepsilon) + O(|V'|^2) + O(|\underset{\sim}{E}|^2) , \\ C'' &= L'' + M'' + O(\varepsilon) + O(|V'|) + O(|\underset{\sim}{E}|^2) , \end{aligned} \tag{5.22}$$

where M' and M'' are square matrices of orders l' and $l - l'$, respectively. According to (5.21), the matrix M is block diagonal, $M = \{M', M''\}$, where M' is arbitrary and M'' commutes with L'':

$$L''M'' = M''L'' . \tag{5.23}$$

$\underset{\sim}{E}$ denotes the collection of all the parameters. Now we shall show that in the set $\mathscr{B}''$ we always have a family of the form

$$\begin{gathered} V' = 0 , \quad \underset{\sim}{K} = \underset{\sim}{\mathring{K}}(\varepsilon) , \quad \underset{\sim}{N}' = \underset{\sim}{\mathring{N}}'(\varepsilon) , \quad M' = \mathring{M}'(\varepsilon) , \quad M'' = \mathring{M}''(\varepsilon) ; \\ \underset{\sim}{\mathring{K}} = 0 , \quad \underset{\sim}{\mathring{N}} = 0 , \quad \mathring{M}' = 0 , \quad \mathring{M}'' = 0 \quad \text{for} \quad \varepsilon = 0 . \end{gathered} \tag{5.24}$$

By (5.22), equations (5.16) and 5.17 (for $a = 1$, $V' = 0$) have the form

$$\begin{aligned} &\underset{\sim}{K} + O(\varepsilon) + O(|\underset{\sim}{E}|^2) = 0 \\ &\underset{\sim}{N}' + O(\varepsilon) + O(|\underset{\sim}{E}|^2) = 0 \end{aligned} \tag{5.25}$$

The matrices (5.18) for $a = 1$ are

$$B_1' = M' + O(\varepsilon) + O(|\underset{\sim}{E}|^2) \ ,$$

$$B'' = M'' + O(\varepsilon) + O(|\underset{\sim}{E}|^2) \ ,$$

and according to (5.14), $L''B'' = B''L''$. According to the Implicit Function Theorem the system of equations (5.25), $B_1' = 0$, $B'' = 0$ has a unique solution of the form (5.24), and condition (5.23) is satisfied. Evidently, this solution lies in the set $\mathscr{B}''$. Thus, theorem 1 in Bibikov's paper [Bibikov, 1973a] follows from theorem 5.4. In the same way one of Moser's results also follows from Theorem 5.4 [Moser, 1967, theorem 1]. In Bibikov's work the quantities Ω, L, A_1, $\underset{\sim}{K}$, $\underset{\sim}{N}$ and M are denoted by ω, Ω, P, λ, μ, and M, respectively.

Author's Comments (1986)

1. Generally speaking, each formal change of coordinates is not unique (i.e. the normalizing transformation, the seminormalizing transformation etc.). So statements of the theorems on analyticity (convergence) of such transformations should be understood as follows: there exists a corresponding transformation, which is analytic (converges)....

2. The sets $\mathscr{A}_W$ and $\mathscr{B}_W$ contain invariant subsets $\mathscr{A}_+$, $\mathscr{A}_-$ and $\mathscr{B}_+$, $\mathscr{B}_-$ respectively (see property 6 of section 1.5). The analyticity of the corresponding invariant sets $\mathscr{A}_+$, $\mathscr{A}_-$ or $\mathscr{B}_+$, $\mathscr{B}_-$ is a consequence of the analyticity of the sets $\mathscr{A}_W$ or $\mathscr{B}_W$. Similarly for sets $\mathscr{A}_W \cap \mathscr{L}_+$, $\mathscr{A}_W \cap \mathscr{L}_-$ and $\mathscr{B}_W \cap \mathscr{L}_+$, $\mathscr{B}_W \cap \mathscr{L}_-$, where $\mathscr{L}$ is a formal invariant subspace (see §§ 4 and 5). The analyticity of those formal invariant sets is the main result of the work. The analyticity of the other parts of the sets $\mathscr{A}_W$ and $\mathscr{B}_W$ is not interesting because they are not invariant. In my subsequent works, I changed the approach to the problem and looked only for analytic invariant sets [see § 3, of Ch. III of the book "Local Method...," Part I in this translation].

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J. Guckenheimer, P. Holmes

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